On the local temperature of a quantum field.

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Contents

Basic Setting

M is a globally hyperbolic Lorentzian manifold with

- background matter satisfying Einstein's Equation

$$T^{\mathrm{bg}}_{\mu\nu} = \frac{1}{8\pi} G_{\mu\nu} = \frac{1}{8\pi}\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right),$$

- a scalar quantum field ϕ satisfying the Klein-Gordon equation

$$K\phi := (-\Box + \xi R + m^2)\phi = 0$$

with mass $m \geq 0$ and scalar curvature coupling $\xi \in \mathbb{R}$.

ϕ is treated as a test-field, so in a general state ω

$$\omega(T^{\mathrm{ren}}_{\mu\nu}(\phi)) + T^{\mathrm{bg}}_{\mu\nu} \neq \frac{1}{8\pi} G_{\mu\nu}.$$

Global Temperature

If M is stationary with preferred time flow χ^μ, we say

$$\omega \text{ is in global thermal equilibrium at temperature } T \geq 0$$
$$\Leftrightarrow$$
$$\omega \text{ satisfies the } \beta\text{-KMS condition with } T = \beta^{-1} \text{ w.r.t. } \chi^\mu.$$

Notation: $\omega^{(\beta)}$, where $\beta = \infty$ denotes a ground state.

- The interpretation of $\omega^{(\beta)}$ as thermal equilibrium states is motivated by analogy with quantum statistical mechanics.
- $\omega^{(\beta)}$ is stationary, i.e. invariant under the time flow χ^μ.
- $\omega^{(\beta)}$ exists for all $\beta \in (0, \infty]$ under suitable circumstances (e.g. $\xi R + m^2 > 0$ everywhere).

Shortcomings of Global Temperature

The β-KMS condition raises some problems:

- T is a global property:
  Temperatures depend on $x \in M$ (e.g. in non-equilibrium thermodynamics), but χ^μ, the β-KMS condition and T are global.
- In curved spaces, the interpretation may be flawed:
  If the background energy density is negative at $x \in M$,

  $$\chi^\mu \chi^\nu T^{\mathrm{bg}}_{\mu\nu}(x) < 0$$

  can ϕ continually transfer energy (and entropy) to the background matter via the metric? Can we trust the thermodynamical interpretation of β-KMS states then?

Local Temperature

For general M, mass $m = 0$, a Hadamard state ω and

$$:\phi^2:\,(x) = \lim_{y \to x} \phi(x)\phi(y) - H(x,y),$$

a locally covariant Wick square, we say

$$\omega \text{ has a local temperature } T_\omega(x) \text{ at } x,$$

$$T_\omega(x) := \sqrt{12\,\omega(:\phi^2:(x))},$$

whenever $\omega(:\phi^2:(x)) \geq 0$. Otherwise, $T_\omega(x)$ is not defined.

This formula was proposed by

- Buchholz, Ojima and Roos, Ann. Physics **297** 219–242 (2002),
- Buchholz and Schlemmer, Class. Quantum Grav. **24**, F25–F31 (2007).

Motivation of the Local Temperature

Buchholz' and Schlemmer's motivation:

- All β-KMS states are Hadamard.
- In Minkowski space,

$$T_{\omega^{(\beta)}}(x) \equiv \beta^{-1} = T.$$

  Thus $:\phi^2:(x)$ is a local thermometer in Minkowski space.
- $:\phi^2:(x)$ is generally covariant, so it is a local thermometer in general M, at least when $m = 0$ (and perhaps $\xi = \frac{1}{6}$).

Remark:
For massive fields the relation between β and $\omega^{(\beta)}(:\phi^2:(x))$ is different. The definition of $T_\omega(x)$ could be modified accordingly.

Measuring Local Temperature

To measure $T_\omega(x)$, consider an Unruh-DeWitt detector on a worldline:

- For a long interaction interval, one finds thermal behaviour, but the temperature is not localised in time.
  Unruh, Phys. Rev. D **14**, 870–892 (1976).
- Requiring detailed balance puts limitations on derivations for short interaction intervals.
  Fewster, Juárez-Aubry and Louko, Class. Q. Grav. **33**, 165003 (2016).
- For stationary states in stationary M, the detector measures

$$T'_\omega(x) = \sqrt{T_\omega(x)^2 + \frac{a_\mu a^\mu - R_{\mu\nu} v^\mu v^\nu}{4\pi^2}} \, ,$$

  which is constant along stationary worldlines.
  Lynch and Afshordi, arXiv:1611.06619.

If acceleration and curvature vanish, the detector agrees with $T_\omega(x)$.

Problems with Local Temperature

$T_\omega(x)$ also raises problems:

- There is a renormalisation ambiguity:

$$:\phi^2:' \ = :\phi^2: + c_1 R + c_2 m^2, \qquad c_1, c_2 \in \mathbb{R}.$$

  $m = 0$, but what is $T_\omega(x)$ when $R(x) \neq 0$?
  Cf. Hollands and Wald, Commun. Math. Phys. **223**, 289–326 (2001).
- Stationary observers may be accelerating and rotating:
  How should the apparent forces on the systems they observe be taken into account? (Cf. Unruh effect.)
- $T_\omega(x)$ is not defined when $\omega(:\phi^2:(x)) < 0$:
  Which states have a local temperature? All states? Ground states?

Quantum Inequalities

$T_\omega(x)$ only exists when $\omega(:\phi^2:(x)) \geq 0$.

- Many states have no local temperature:

$$\omega(:\phi^2:(x))$$

  is unbounded from below as ω ranges over all Hadamard states.
- Quantum inequalities:
  Suitable time averages of $\omega(:\phi^2:(x))$ give a finite lower bound independent of ω. In analogy to time-energy uncertainty,

  $\omega(:\phi^2:(x))$ cannot be too negative for too long.

  However,
  - the lower bound may be negative,
  - the bound is not point-wise on $\omega(:\phi^2:(x))$.
- Even ground states may have negative $\omega^{(\infty)}(:\phi^2:(x))$.

Ground States Without Local Temperature I

If M is stationary and ω a stationary state (e.g. the ground state), then

$$\tau \mapsto \omega(:\phi^2:(\gamma(\tau)))$$

is constant along the stationary worldline $\tau \mapsto \gamma(\tau)$.

$\omega(:\phi^2:(\gamma(\tau)))$ can be negative "forever":

- The Minkowski vacuum $\omega^{(\infty)}$ restricted to the Rindler spacetime is a $\frac{1}{2\pi}$-KMS state. (Unruh, Phys. Rev. D **14** 870–892 (1976).)
- $\omega^{(\infty)}(:\phi^2:(x)) \equiv 0$ and $T_{\omega^{(\infty)}}(x) \equiv 0$.
- The Fulling vacuum ω^F in Rindler spacetime has

$$\omega^F(:\phi^2:(x)) < \omega^{(\infty)}(:\phi^2:(x)) = 0.$$

  It has no local temperature (and negative energy-density).

Cause: accelerated stationary observers.

Energy Conditions

Gravity is attractive, so $T^{\rm bg}_{\mu\nu}$ should satisfy for timelike v^μ e.g.

$$\begin{aligned} v^\mu v^\nu T^{\rm bg}_{\mu\nu} &\geq 0 \qquad \text{(weak energy condition)},\\ v^\mu v^\nu R_{\mu\nu} &\geq 0 \qquad \text{(strong energy condition)}. \end{aligned}$$

If M is stationary and the stationary observers are not accelerated:

- M is ultra-static

$$M = \mathbb{R} \times \Sigma, \qquad g = -dt^2 + h_{ij}(x)dx^i dx^j.$$

- $R_{0\mu} = 0$ and the weak/strong/dominant energy conditions all are

$$R_{\mu\nu} \geq 0.$$

In particular, $R = g^{\mu\nu}R_{\mu\nu} = h^{ij}R_{ij} \geq 0$.

Ground States Without Local Temperature II

$R \not\geq 0$ can lead to ground states without local temperature:

- Choose $M = \mathbb{R}^4$ with

$$g = -dt^2 + \Omega^2(x)\delta_{ij}dx^i dx^j .$$

- Choose Ω such that

$\Omega \geq 1$	(M is globally hyperbolic),
$\Omega \equiv 1$ near $x = 0$	(M is locally Minkowski space),
$\Delta(\ln \Omega) \geq 0$ is non-trivial	($R \leq 0$ is non-trivial),
Ω is bounded	($\omega^{(\infty)}$ exists for $m = \xi = 0$).

Such Ω exist and on some open $O \subset M$

$$\omega^{(\infty)}(:\phi^2:) < 0, \qquad \Omega \equiv 1.$$

Cause: ~~acceleration~~, ~~local curvature~~, violation of energy conditions.

Existence of Local Temperature

These are sufficient conditions for the existence of $T_\omega(x)$:

Theorem

Assume:

- *M is ultra-static with a compact Cauchy surface Σ and scalar curvature $R \geq 0$ non-trivial.*
- *The Riemann curvature vanishes on an open set $O \subset \Sigma$.*
- *ϕ has $m = 0$ and $\xi \in (0, \frac{1}{6})$.*

Then $T_\omega(x)$ exists for all $x \in O$ and all stationary Hadamard states ω.

Remark:
Near O, M is Minkowski space, so no local physics enters.

The Class of Spacetimes

To find spacetimes satisfying the assumptions we need:

- a compact Cauchy surface (Σ, h) with
- non-trivial $R \geq 0$ and
- a flat open region $O \subset \Sigma$.

Example

Embed $\mathbb{S}^3$ into Euclidean $\mathbb{R}^4$ and flatten its top, keeping it convex. Then choose h the induced metric.

One can construct other examples by

- taking small perturbations of $g_{\mu\nu}$ in regions where $R > 0$,
- using gluing techniques.
  Cf. Delay Differential Geom. Appl. **29** (2011), 433–439.

Properties of Ground States

Suppose M is stationary and a ground state $\omega^{(\infty)}$ exists.

Proposition

For every stationary Hadamard state ω we have

$$\omega(:\phi^2:(x)) \geq \omega^{(\infty)}(:\phi^2:(x)).$$

for all $x \in M$. Moreover, for $\omega_2(x, y) := \omega(\phi(x)\phi(y))$

$$\omega_2(x, y) - \omega_2^{(\infty)}(x, y)$$

is a smooth function of positive type on $M \times M$.

Conclusion: If $T_{\omega^{(\infty)}}(x)$ exists, so does $T_\omega(x)$ for all stationary ω.

Properties of Local Temperature

Suppose M is stationary and β-KMS states exist for all $\beta > 0$.

Proposition

The map

$$T \mapsto \omega^{(\beta)}(:\phi^2:(x)), \qquad \beta = T^{-1}$$

is continuous and monotonically increasing in $T \geq 0$. *Moreover,*

$$\omega_2^{(\beta)}(x, y) - \omega_2^{(\beta')}(x, y)$$

is a smooth function of positive type on $M \times M$ *if* $\beta < \beta'$.

Conclusion: When local and global temperature both make sense,

$$T \mapsto T_{\omega^{(\beta)}}(x), \qquad T = \beta^{-1}$$

is continuous and monotonically increasing.

Proving the Existence Theorem

1. If $\omega^{(\beta)}(:\phi^2:(x)) \geq 0$ for all $\beta \in (0,\infty)$ then

$$\omega(:\phi^2:(x)) \geq \omega^{(\infty)}(:\phi^2:(x)) = \lim_{\beta\to\infty} \omega^{(\beta)}(:\phi^2:(x)) \geq 0$$

   and $T_\omega(x)$ exists for all stationary ω.

2. $M = \mathbb{R} \times \Sigma$ is ultra-static, so we can use a Wick rotation:

$$\tilde{M}_\beta := \mathbb{S}^1_\beta \times \Sigma\,, \qquad \tilde{g} = d\tau^2 + h$$

   and for $y = (0, q) \in M$, $\tilde{y} = (0, q) \in \tilde{M}$,

$$\omega^{(\beta)}(:\phi^2:(y)) = \lim_{\tilde{x}\to\tilde{y}} \left(\tilde{G} - \tilde{H}\right)(\tilde{x}, \tilde{y}),$$

   with $\tilde{G} = (-\Delta_{\tilde{g}} + \xi R)^{-1}$ and $\tilde{H}$ the Hadamard parametrix near $\tilde{x}$.

Note: $R \geq 0$ is non-trivial, so $\tilde{G}$ exists and is C^∞ on $\tilde{x} \neq \tilde{y}$.

Proving the Existence Theorem

3. Fix $y = (0, q) \in O \subset M$ with $\tilde{y} = (0, q) \in \tilde{M}$.

$$\Omega(\tilde{x}) := 4\pi^2 \tilde{G}(\tilde{x}, \tilde{y}) > 0$$

on $\tilde{M} \setminus \{\tilde{y}\}$ (strong maximum principle). Near $\tilde{y}$

$$\Omega(\tilde{x}) = |\tilde{x}|^{-2} + 4\pi^2 \omega^{(\beta)}(:\phi^2:(y)) + \dots$$

4. $\tilde{M}_\beta \setminus \{\tilde{y}\}$ is asymptotically flat for

$$\hat{g} := \Omega^2 \tilde{g}$$

with $\tilde{y} =$ at infinity and ADM mass $\omega^{(\beta)}(:\phi^2:(y))$.

R. Schoen J. Differential Geom. **20** (1984) 479–495.

Proving the Existence Theorem

5. Using $(-\Delta_{\tilde{g}} + \xi R)\Omega = 0$ the scalar curvature of $\hat{g}$ is

$$\hat{R} = \Omega^{-2}(R - 6\Omega^{-1}\Delta_{\tilde{g}}\Omega) = (1 - 6\xi)\Omega^{-2}R \geq 0.$$

By the positive mass theorem in 4 dimensions

$$\omega^{(\beta)}(:\phi^2:(y)) \geq 0.$$

R. Schoen and S.-T. Yau Phys. Rev. Lett. **42**, 547–548 (1979).

Conclusions

Local and global temperature give qualitatively similar information:

- If M is stationary
$$T = \beta^{-1} \mapsto T_{\omega(\beta)}(x)$$
is continuous and monotonic as long as $T_{\omega(\beta)}(x)$ exists.
- $T_{\omega}(x)$ exists under physically reasonable conditions: in ultra-static M, compact Σ, $R \geq 0$ non-trivial, $m = 0$, and $\xi \in (0, \frac{1}{6})$ for stationary ω on flat regions.

This follows from corresponding results on $\omega(:\phi^2:(x))$.

$T_{\omega(\infty)}(x)$ may fail to exist due to

- accelerated observers (Unruh effect),
- violation of energy conditions (ultra-static M with $R \not\geq 0$).

Outlook

Opportunities for extensions:

- include acceleration/non-zero curvature (cf. Lynch-Afshordi),
- allow non-compact Cauchy surfaces (with asymptotic flatness or stronger curvature conditions),
- other local observables, e.g. the renormalised stress tensor $T^{\mathrm{ren}}_{\mu\nu}$,
- massive theories.

Clarify relations to

- black hole thermodynamics/Hawking temperature,
- energy conditions of background matter,
- semi-classical Einstein equation.