Hadamard states from light-like hypersurfaces

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Summary

References

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- C. Dappiaggi, V. M., N. Pinamonti: Rigorous Steps towards Holography in Asymptotically Flat Spacetimes RMP 2006

Technical Reference (*-algebras, states, etc.)

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1. Free AQFT in fixed curved spacetime

- Fields are **quantized** but they interact with **classical** gravity.
- Backreaction on the geometry can be *a posteriori evaluated* through Einstein equations $\langle \hat{T}_{ab}(x) \rangle = G_{ab}(x)$.
- No **general** locality and covariance issues (spacetime fixed).

1.1. Classical Structures: Global Hyperbolicity

- Good (4D smooth) spacetimes (M, g) are those **initial data** over suitable smooth spacelike 3-surfaces Σ **uniquely** determine solutions of **field equations**.
- **(Time oriented) globally hyperbolic** spacetimes [Wa84] satisfy the requirement. The said smooth spacelike 3-surfaces are **Cauchy surfaces** and field equations of **hyperbolic type** constructed out of the metric g
- **Klein Gordon** is the simplest case of **linear** hyperbolic equation for a smooth field $\phi : M \to \mathbb{R}$

$$P\phi := (\Box_g - m^2 + \xi R)\phi = 0 \qquad \textbf{(KG)}$$

- Given smooth compactly supported **Cauchy data** over Σ

$$\phi|_\Sigma \,, \quad \partial_{N_\Sigma}\phi|_\Sigma$$

$\Rightarrow$ unique smooth solution of **(KG)** everywhere in M.

1.2. Classical Structures: Fundamental Solutions

- **Retarded** and **advanced fundamental solutions** exist [BGP07]

$$G_\pm : C_0^\infty(M) \to C^\infty(M) \quad \textbf{linear}$$

Associating (real) smooth compactly supported sources f with the **unique** solution $\phi_f = G_\pm(f)$ of

$$P\phi_f = f$$

such that $G_\pm(f)$ is resp. **supported** in $J_\pm(supp(f))$ (the **causal future/past** of $supp(f)$).

- If $S(M) := \{\phi : M \to \mathbb{R} \,|\, \text{solves (KG), compact Cauchy data }\}$ then the linear map

$$G := G_- - G_+ : C_0^\infty(M) \to S(M)$$

is **surjective** and $h \in Ker(G) \Leftrightarrow h = Pf$ for some $f \in C_0^\infty(M)$.

1.3. Classical Structures: Symplectic structures

- The space of solutions $S(M)$ has a **symplectic form**

$$\sigma_M(\phi, \phi') = \int_\Sigma \phi \nabla_{N_\Sigma} \phi' - \phi' \nabla_{N_\Sigma} \phi \ d\Sigma_g$$

  σ is **antisymmetric**, Cauchy surface (Σ) **independent**

- σ_M is **weakly non-degenerate** ($\Leftarrow$ well-posed Cauchy prob.):

  $\sigma_M(\phi, \phi') = 0$ for all $\phi' \in S(M)$ implies $\phi = 0$.

- $G_\pm$, G are suitably **continuous** $\Rightarrow$ **distributional kernels** exist

$$G(f)(x) = \int_M G(x, y) f(y) d\mu_g(y)$$

  From Poincaré-Stokes' theorem, for $f, f' \in C_0^\infty(M)$,

$$\sigma_M(G(f), G(f')) = \int_{M^2} G(x, y) f(x) f'(y) d\mu_g(x) d\mu_g(y)$$

1.4. Quantum Structures: *-algebra of fields

- **Algebraically quantising** ϕ over a **given** spacetime (M, g) means associating ϕ with an abstract **unital** ***-algebra** $\mathcal{A}_\Phi(M)$. $\mathcal{A}_\Phi(M)$ is made of all finite **complex** linear combinations of finite products of the unit $\mathbb{I}$ and **(smeared quantized) field operator** $\Phi(f)$ with $f \in C_0^\infty(M)$.

Specific Requirements

- 1.**Linearity** $\Phi : C_0^\infty(M) \to \mathcal{A}(M)$ is $\mathbb{R}$-linear.
- 2.**Hermiticity** $\Phi(f)^* = \Phi(f)$.
- 3.**Com.Rel.** $[\Phi(f), \Phi(f')] = iG(f, f')$ $(= i\sigma_M(G(f), G(f')))$.
- 4.**Field equations** $\Phi(P(f)) = 0$ (distributional sense)

Remark: (3) represents causality and CCR simultaneously
(a) Def of $G \Rightarrow [\Phi(f), \Phi(f')] = 0$ if $supp(f)$ and $supp(f')$ are **causally separated**: uncorrelated measurement
(b) Interplay of G & $\sigma_M \Rightarrow [\Phi(t, x), \nabla_{N_\Sigma}\Phi(t, y)] = i\delta(x, y)$

1.5. Quantum Structures: states

No **preferred** *quantum vacuum state / quantization procedure* in a generic (M, g). The **algebraic notion** of state is convenient.

- A (quantum) state on a (complex) unital *-algebra $\mathcal{A}$ is a map $\omega : \mathcal{A} \to \mathbb{C}$ that is **linear**, **positive** ($\omega(a^*a) \geq 0$), and **normalised** ($\omega(\mathbb{I}) = 1$).
- $\omega(a)$, for $a = a^*$ has the meaning of **expectation value** of the **observable** a in the state ω.
- **GNS construction**. Given a state $\omega : \mathcal{A} \to \mathbb{C}$, there exist
  (1) a Hilbert space $\mathcal{H}_\omega$,
  (2) a dense subspace $D_\omega \subset \mathcal{H}$,
  (3) a unital *-algebra representation $\Pi_\omega : \mathcal{A} \to \mathcal{L}(D_\omega)$
  (4) a unit vector $\Psi_\omega \in D_\omega$ such that

  $$D_\omega = \Pi_\omega(\mathcal{A})\Psi_\omega \quad \text{and} \quad \omega(a) = \langle \Psi_\omega | \Pi_\omega(a)\Psi_\omega \rangle_\omega \, .$$

  $(\mathcal{H}_\omega, D_\omega, \Pi_\omega, \Psi_\omega)$ is determined by ω up to **unitary** transformations.

1.6. Quantum Structures: Gaussian/Quasifree states

- If $\mathcal{A} = \mathcal{A}_\Phi(M)$ and ω its **two-point function** is $\omega(\Phi(f)\Phi(h))$.
- **Gaussian/Quasifree states**: states with
  - $\omega(\Phi(f)) = 0$
  - $\omega(\Phi(f_1)\cdots\Phi(f_n))$ from $\omega(\Phi(f_h)\Phi(f_k))$ via Wick's rule.
- GNS costruction for Gaussian states:
  - $\mathcal{H}_\omega = F_+(\mathcal{H}_\omega^{(1part)})$ Bosonic Fock space,
  - Ψ_ω Fock vacuum
  - $\Pi_\omega(\Phi(f)) = \hat{\Phi}(f) = a_{V_\omega \phi_f} + a^\dagger_{V_\omega \phi_f}, \quad \phi_f = G(f) \in S(M)$
  - $V_\omega : S(M) \to \mathcal{H}_\omega^{(1part)}$ ($\mathbb{R}$-linear, dense complexified range).
- **Example**: (M, g) with a **timelike Killing vector field** K e.g. Minkowski st, Schwarzschild wedges, part of de Sitter st etc. $\Rightarrow \exists$ (unique under mild hypotheses [Ka78,KW91]) Gaussian state ω_K such that

$$V_{\omega_K}(\phi) = \phi_+ \text{ positive-frequence part of } \phi = \phi_+ + \phi_-$$

NB frequence **referred to the Killing time**.

1.7.1. Quantum Structures: Hadamard states

- Meaningful states must be used to compute the back reaction on the metric through Einstein equations, **enlarging** $\mathcal{A}_\Phi(M)$ with elements $T_{\mu\nu}(x)$ [Wa78,Mo03,BFV03] and in perturbative renormalization involving also $\phi^n(x)$ and $T(\phi^{n_1}(x_1)\cdots\phi^{n_k}(x_k))$ [BF00,BFK95,HW01-04,BDF09,KM16]
- **Hadamard states do the job**: [FSW78,FNW81,KW91] Gaussian states

$$\omega(\Phi(f)\Phi(h)) = \int_{M^2} \omega_2(x,y)f(x)h(y)d\mu_g(x)d\mu_g(y)$$

short distance singularity similar to that of Minkowski vacuum

$$\omega_2(x,y) = \text{w-}\lim_{\epsilon\to 0^+} \frac{u(x,y)}{\sigma_\epsilon(x,y)} + v(x,y)\ln\sigma_\epsilon(x,y) + w_\omega(x,y)$$

- ◦ $\sigma_\epsilon(x,y)$ (regularized) **squared geodesical distance**,
- ◦ u, v **universal functions** of **local geometry**,
- ◦ w_ω determined by ω

1.7.2. Quantum Structures: Hadamard states

- **Equivalent** definition of Hadamard states in terms of Duistermaat-Hörmander's wave front set of $\omega_2(x, y)$.
- If $f \in D'(M)$ (distribution), the **wave front set** $WF(f) \subset T^*M$ consists of the pairs $x \in M$, $p \in T^*_p M \setminus \{0\}$ such that

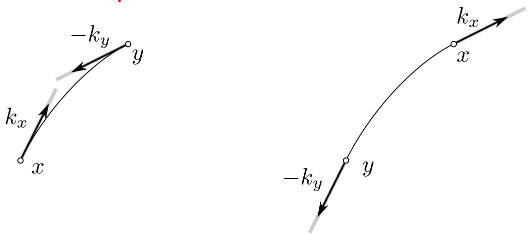
  the "Fourier trnsm at x" of f does not vanish rapidly along p.
- The distribution f is **not** a smooth function about x if $(x, p) \in WF(f)$ in particular.
- **Example**: $WF(\delta_{x_0}) = \{(x_0, p) \mid 0 \neq p \in T^*_{x_0} M\}$
- WF useful in many contexts (propagation of singularities in PDE theory, criteria for multiplying distributions or restricting them on submanifolds, WFs can be composed, etc.)
- Referring to distributions $K \in D(M \times M)'$ as two-point functions, $WF(K) \subset T^*(M \times M)$

THEOREM [Ra96a,Ra96b] *A Gaussian state* $\omega : \mathcal{A}(M) \to \mathbb{C}$ *with* $\omega_2 \in D'(M \times M)$ *is Hadamard iff*

$$WF(\omega_2) = \{(x, y, k_x, -k_y) \in T^*(M \times M) \setminus 0 \mid (x, k_x) \sim (y, k_y)\,, k_x \triangleright 0\}$$

∘ $(x, p) \sim (y, q)$ *means there is a lightlike geodesic through* x *and* y *with co-tangent vectors* p *at* x *and* q *at* y.

∘ $p \triangleright 0$ *means that* p *is future-oriented.*



- The arising Feynman propagator is coherent with the popular statement that *it propagates positive frequencies towards positive times and negative ones backward in time.*

1.7.4. Quantum Structures: Hadamard states

- Minkowski vacuum and several concrete natural Gaussian isometry-invariant states of symmetric spacetimes are Hadamard. Time-invariant Gaussian states in static globally hyperbolic spacetimes are Hadamard [FNW81,SV00].
- Hadamard states exist in generic globally hyperbolic spacetime (proofs based on the original deformation argument [FNW81])
- States related with Black-Hole physics and Hawking radiantion have been proved to be Hadamard [DMP11,Sa13]
- From Hadamard property important results in relation to KMS condition black hole radiation [FH89,KW91,MP12,CMP14].
- Hadamard states arise in cosmological models (FRW) (e.g. low-energy states) [Ol07,DHP11,TB13,Av14].
- Important role in direct approaches to semiclassical Quantum Gravity [Pi11]
- Alternate candidates [AAS12] of physically meaningful states, but Hadamard condition remains crucial [FV12,BF13,FV13].

2. Null 3-surfaces and bulk-boundary algebra correspondence

- The general problem is defining Hadamard states for some types of physically meaningful spacetimes (M, g) admitting lightlike completion I, with a (common) geometric structure.

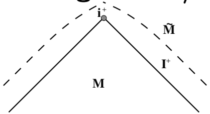
- We define a (common) *-algebra of observables $\mathcal{A}(I)$ on I such that the algebra of fields in the bulk $\mathcal{A}_\phi(M)$ embeds into $\mathcal{A}(I)$.

- Defining states on $\mathcal{A}(I)$ we induce back states on $\mathcal{A}_\phi(M)$.

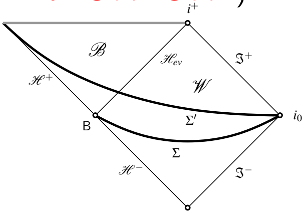
2.1. Relevant spacetimes and boundary structures

We focus of some glob.hyp. spacetimes (M, g) admitting a (possibly conformal) **boundary** $\Im^{\pm}$ made of **light-like 3-surface(s)** in a larger (possibly unphysical) spacetime $(\widetilde{M}, \widetilde{g})$.

- Asymptotically flat spacetimes at future/past null infinity
- Cosmological spatially-flat FRW models admitting a **common cosmological event/particle horizon** (es. dS spacetime) and **non-homogeneous/non-isotropic** deformations.



- Part of Kruskal manifold (**BH** region + right **Schwarzschild wedge** $\Im^- = \mathfrak{I}^- \cup \mathcal{H}^- \cup \mathcal{H}^+$)



2.2.1 Geometry of $\widetilde{M}$ and $\mathscr{I}$: metric

- The metric $\tilde{g}$ over $\widetilde{M}$ is and extension of g over M or a conformal (singular) extension of g: $\tilde{g} := \Omega^2 g$.
- $\tilde{g}$ is **degenerate** on $\mathscr{I}$ and around $\mathscr{I}$ takes the complete Bondi's form

$$\tilde{g} = c(-da \otimes du - du \otimes da + d\mathbb{S}^2)$$

  - $u \in \mathbb{R}$ affine parameter of null geodesics forming $\mathscr{I}$,
  - a transverse coordinate (possibly $a = \Omega$). $a = 0$ exactly on $\mathscr{I}$,
  - $d\mathbb{S}^2$ standard metric on $\mathbb{S}^2$, $c > 0$ constant.
- $\mathscr{I}$ equipped with the degenerate metric h and $n = \partial_u$ admits an **infinite-dimensional** (non-locally-compact, topological) **group of diffeomorphisms** $G_{\mathscr{I}} \ni g : \mathscr{I} \to \mathscr{I}$, preserving physically meaningful geometrical structures.
- $G_{\mathscr{I}}$ = **BMS group** (e.g., [Wa84]) for **asymptotically flat spacetimes** or another infinite-dimensional group [DMP09] for **cosmological models**, u is related to conformal time.

2.2.2 Geometry of $\widetilde{M}$ and $\Im$: symmetries and symplectic structure

THEOREM [AX78,DMP09]
Let ξ *be a* **Killing vector** *of* (M, g), *then*
(a) it smoothly extends to a **(conformal) Killing vector** *on* $(\widetilde{M}, \widetilde{g})$ *tangent to* $\Im$,
(b) the generated one-parameter group of diffeomorphisms of $\Im$ *is a one-parameter* **subgroup** *of* $G_\Im$.

- The vector space $S(\Im)$ of regular maps $\psi : \Im \to \mathbb{R}$ rapidly vanishing at infinity admits a **weakly-nondegenerate symplectic form**

$$\sigma_\Im(\psi, \psi') = \int_\Im \psi' \partial_u \psi - \psi \partial_u \psi' \ du \wedge \mu_{\mathbb{S}^2}$$

- $S(\Im)$ and $\sigma_\Im$ are $G_\Im$-**invariant**: $\sigma_\Im(g_* \psi, g_* \psi') = \sigma_\Im(\psi, \psi')$ where $g \in G_\Im$ and $g_* \psi := \psi \circ g^{-1} \in S(\Im)$ if $\psi \in S(\Im)$.

2.2.3 Geometry of $\widetilde{M}$ and I: bulk-boundary symplectic injection

If some natural hypotheses are satisfyed for the spacetime (M, g) embedded in $(\widetilde{M}, \widetilde{g})$, in particular

- If M is Kruskal spacetime/an asymptotically flat spacetime. $P = \Box_g - \frac{1}{6}R$ (i.e. $m = 0$ and **conformal coupling**),
- If M is an asymptotically flat spacetime also i^+/i^- exist,

then $\phi \in S(M)$ **uniquely** and **smoothly extends** to some $h_\phi \in S(I)$ (a **singular** conformal transformation may be involved).

THEOREM [DMP06,M06,DMP08,DMP09,DMP11]
The map $\Gamma : S(M) \ni \phi \mapsto h_\phi \in S(I)$ *is linear, injective, and preserves the symplectic forms*

$$\sigma_M(\phi, \phi') = \sigma_I(\Gamma(\phi), \Gamma(\phi'))$$

Remark. $S_\Phi(M)$ depends on the **concrete** spacetime. $S(I)$ is **in common** with the class.

2.3.1 The boundary *-algebra $\mathcal{A}(I)$ and states: algebras

Referring to the **symplectic space** $(S(I), \sigma_I)$, we define an abstract unital *-algebra $\mathcal{A}(I)$ with generators $\Psi(h)$ satisfying

- 1.**Linearity** $\Psi : S(I) \to \mathcal{A}(M)$ is $\mathbb{R}$-linear.
- 2.**Hermiticity** $\Psi(h)^* = \Psi(h)$.
- 3.**Com.Rel.** $[\Psi(h), \Psi(h')] = i\sigma_I(h, h')$.

Since Γ is **injective** and **preserves the syplectic forms**:

THEOREM [Mo08,DMP08,DMP09]

There exists a unique unital *-**algebra injective homomorphism** $\imath_M : \mathcal{A}_\phi(M) \to \mathcal{A}(I)$ *such that*

$$\imath_M(\Phi(f)) = \Psi(\Gamma(G(f)))$$

If $\omega : \mathcal{A}(I) \to \mathbb{C}$ *is a* **state**, $\omega_M := \omega \circ \imath_M$ *is a* **state** *on* $\mathcal{A}_\phi(M)$.

2.3.2 The boundary *-algebra $\mathcal{A}(I)$ and states: symmetric states

THEOREM [Mo08,DMP08,DMP09]

For every **one-parameter** ***-algebra group of automorphisms**

$$\alpha_t^{(\xi)} : \mathcal{A}_\phi(M) \to \mathcal{A}_\phi(M)$$

induced by a Killing vector ξ *of* M *there is* **one-parameter** ***-algebra group of automorphisms**

$$\beta_t^{(\xi)} : \mathcal{A}(I) \to \mathcal{A}(I)$$

induced by a corresponding one-parameter subgroup of G_I*, such that*

$$\imath \circ \alpha_t^{(\xi)} = \beta_t^{(\xi)} \circ \imath$$

2.3.3 The boundary *-algebra $\mathcal{A}(I)$ and states: symmetric states

- If $\omega : \mathcal{A}(I) \to \mathbb{C}$ is **invariant** under G_I, then ω_M is invariant under the action of **every Killing isometry** (if any) of (M, g).

$$\omega_M(\alpha_t^{(\xi)}(a)) = \omega_M(a) \quad \forall a \in \mathcal{A}_\phi(M)$$

- Passing to the **Hilbert-space** representation via **GNS theorem**:

  ◦ $\alpha_t^{(\xi)}$ is **implementable** with a **unitary** one-parameter group $U_t^{(\xi)} : \mathcal{H}_\omega \to \mathcal{H}_\omega$

$$\Pi_\omega(\alpha_t^{(\xi)}(a)) = U_t^{(\xi)} \Pi_\omega(a) U_t^{(\xi)*} \, .$$

  ◦ Ψ_ω is **invariant** under $U_t^{(\xi)}$

$$U_t^{(\xi)} \Psi_\omega = \Psi_\omega \, .$$

3. Induced Hadamard Killing-symmetric states

- Is it possible to fix the bondary state ω_I on $\mathcal{A}(I)$ so that the induced bulk state ω_M is **Hadamard**?
- Is it possible to pick out ω_I which is **also** G_I**-invariant**? (So that ω_M is **Killing invariant** in M if Killing symmetries exist.)

3.1.1 Asymptotically flat spacetimes and expanding cosmological models

Main idea: Since $\Im$ admits a $G_\Im$-**invariant** expression of the metric (complete Bondi's form)

$$\widetilde{g} = c(-da \otimes du - du \otimes da + d\mathbb{S}^2)$$

it seems natural to define $\omega_\Im$ decomposing $h \in S(\Im)$ into **positive** and **negative frequencies** $h = h^+ + h^-$ referring to u-Fourier transform and next requiring that

$$\omega_\Im(\Psi(h)\Psi(h')) = \langle h^+ | h'^+ \rangle \quad \forall h, h' \in S(\Im) \qquad \textbf{(S)}$$

THEOREM [MO06,MO09,DMP08,DMP09]
The Gaussian state over $\mathcal{A}(\Im)$ *in* **(S)** *is the* **unique** $G_\Im$-**invariant pure** *state.*
The induced state $\omega_M = \omega_\Im \circ \imath_M$ *over* $\mathcal{A}_\phi(M)$ *is* **Killing-invariant** *and* **Hadamard**.

3.1.2 Asymptotically flat spacetimes and expanding cosmological models

- The proof of Hadamard property obtained by checking $WF(\omega_{M2})$ and using **theorems on composition** of WF sets

  $$\omega_{M2}(f, f') = \omega_I(\Gamma(G(f')), \Gamma(G(f'))) = (\omega_I \circ \Gamma \otimes \Gamma \circ G \otimes G)(f \otimes f')$$

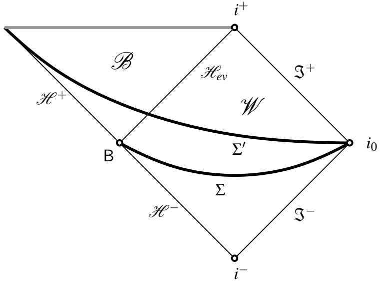
- The precise definition of $S(I)$ plays a crucial role: **regularity** of $h \in S(I)$ and **behaviour** of $h \in S(I)$ for $u \to \pm\infty$.

- In the know very symmetric cases ω_M results to be the natural state:
  ◦ **Poincaré invariant vacuum** in **Minkowski spacetime**,
  ◦ **Bunch-Davies vacuum** in **deSitter spacetime**.

- In cosmological models it is possible to induce states ω_M which are approximated KMS (thermal) states with respect to the conformal time [DHP11].

3.2.1 Unruh state with Hadamard property: ω_I

Unruh state ω_U describes Hawking radiation, detected at $\mathcal{J}^+$, on a spacetime M made of one-half of Kruskal manifold and seems Minkowski vacuum near $\mathcal{J}^-$. The challenge is rigorously defining ω_U proving that it is Hadamard. We consider the **massles** case.



We start by defining ω_I on $I = \mathcal{J}^- \cup \mathcal{H}$ where $\mathcal{H} = \mathcal{H}^- \cup \mathcal{H}^+$ with

- $\widetilde{g} = 4M(-d\Omega \otimes dU - dU \otimes d\Omega + d\mathbb{S}^2)\,, \quad \Omega = 2V$ on $\mathcal{H}$
- $\widetilde{g} = (-d\Omega \otimes dv - dv \otimes d\Omega + d\mathbb{S}^2)\,, \quad \Omega = -2/u$ on $\mathcal{J}^-$

Finally, $\omega_U := \omega_M$ (the state induced on $\mathcal{A}_\phi(M)$ by ω_I).

3.2.2 Unruh state with Hadamard property: ω_U

Many techncal problems arise in defining the space $S(I)$ since the structure about i^- is very complicated and dacay of wavefunctions close to i^0 difficult to study. Results [DMP11]

- ω_I turns out to be thermal (KMS) on $\mathcal{H}$ with respect to ∂_u restriction to ∂_τ (Schwarzschild time) thereon, with $T = T_{Hawking}$.

- ω_I is identical to the boundary state inducing Minkowski vacuum on $\mathcal{I}^-$ and $\omega_U(\phi(x)\phi(y))$ tends to Minkowski 2-point function approaching $\mathcal{I}^-$.

- The induced state ω_U is Hadamard on M and invariant under all the Killing isometries of M.

- A known result [FH89] implies that, as ω_U is Hadamard on $\mathcal{H}_{ev}$, Hawking radiation is detected approacing $\mathcal{I}^+$, examining $\omega_U(\phi(x)\phi(y))$ pushed close to $\mathcal{I}^+$ by the Killing flow of ∂_t.

$+\infty$. Some final comments

- Some of the presented ideas have been developed further, enriched with other ideas, also from alternative viewpoints, during recent years by several authors with applications to Minkowski spacetime, deSitter spacetime and cosmological spacetimes (adiabatic states) [GW14,VW15,GOW16,Va16].
- The embedding $\Gamma : S_\phi(M) \to S(I)$ is **not** surjective. Restricting $S(I)$ to obtain surjectivity is a technically difficult problem solved in [GW16]. It is equivalent to solve the characteristic Cauchy-Goursat problem for KG equation. As a byproduct ω_M was proved to be a pure state as ω_I is.
- A rigorous **existence** proof of the Hartle-Hawking-Israel state establishing also the validity of Hadamard condition has been obtained with a different (yet microlocal) technology [Sa13] (**Uniqueness** known from [KW91].)
- Boundary-bulk induction procedure used in Schwarzshild deSitter spacetime to construct Hadamard states [BJ15].

Thank you very much for your attention!