# Amplitudes on plane waves

L.J.Mason

The Mathematical Institute, Oxford
lmason@maths.ox.ac.uk

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Joint work with Adamo, Casali & Nekovar 2017-8,
arxiv:1706.08925, 1708.09249,

# Motivation

Goal: calculate tree scattering amplitudes on plane-wave space-time backgrounds.

- A first step towards interacting perturbative QFT on curved backgrounds with external particles.
- Plane waves give universal leading order features of gravitational waves via Penrose limit.
- Plane waves satisfy Huygens for scalar wave equation.
- Separable Hamilton-Jacobi equs and linear field equs.
- Extend ambi-twistor-strings to curved backgrounds.
- Test double copy:
  Gravity scattering = Double copy of Yang-Mills scattering.

Focus is on concrete formulae rather than rigour.

# Yang-Mills amplitudes & colour-kinematic duality

Scatter $n$ plane waves of momentum $k_\mu$, polarization $\epsilon_\mu$

$$A_\mu(x) = \epsilon_\mu \mathrm{e}^{ik\cdot x} t^a \,, \qquad k^2 = 0\,, \quad k\cdot\epsilon = 0\,, \quad t^a \in \text{Lie } G.$$

- A YM amplitude is a function $\mathcal{A}(k_i, \epsilon_i, t_i)$, $i = 1, \ldots, n$.
- Suppose that it arises from trivalent Feynman diagrams

$$\mathcal{A} = \sum_\Gamma \frac{N_\Gamma(k_i, \epsilon_i) C_\Gamma(t_i)}{D_\Gamma}\,, \qquad \Gamma \in \{\text{ trivalent diagrams, n legs}\}\,.$$

- $N_\Gamma =$ *kinematic factors*: polynomials in $k_i$, linear in each $\epsilon_i$.
- $D_\Gamma = \prod_{propagators\ e\in\Gamma} (\sum_{i\in e} k_i)^2 =$ *denominators*.
- $C_\Gamma(t_i) =$*colour factor* = contract structure contants at each vertex together along propagators and with $t_i$ at $i$th leg.

## Definition

*The $N_\Gamma$ are said to BCJ numerators if $N_\Gamma$ satisfy identities when $C_\Gamma$ does via Jacobi identities: $C_{\tilde\Gamma} = C_\Gamma + C_{\Gamma'} \Rightarrow N_{\tilde\Gamma} = N_\Gamma + N_{\Gamma'}$.*

Possible at tree-level and up to 4-loops, but not canonical.

# Gravity as double copy of Yang-Mills

Zvi Bern, J J Carrasco, H Johansson, 2008

Scatter $n$ gravity plane waves

$$h_{\mu\nu} = \epsilon_{(\mu}\epsilon_{\nu)}\mathrm{e}^{ik\cdot x}$$

Given BCJ numerators $N_\Gamma$, the gravity tree-amplitude/loop integrand can be obtained as a *double copy* of YM amplitude

$$\mathcal{M}(k_i, \epsilon_i, \epsilon_i) = \sum_\Gamma \frac{N_\Gamma(k_i, \epsilon_i)N_\Gamma(k_i, \epsilon_i)}{D_\Gamma}$$

- First conjectured from string amplitude formulae and KLT relations between YM and gravity amplitudes.
- Proved up to 4-loops.
- There are many extensions to supersymmetric theories.
- Genuine tool for constructing gravity amplitudes.
- No nonperturbative or space-time explanation.

# The three point amplitude

At three points, there is just one trivalent diagram

$$\mathcal{A} = (\epsilon_1 \cdot \epsilon_2 \epsilon_3 \cdot (k_1 - k_2) + \circlearrowleft) f_{abc} t_1^a t_2^b t_3^c = N_{\curlywedge}(\epsilon_i, k_i) C_{\curlywedge}(t_i)$$

It is straightforward to check that for gravity

$$\mathcal{M} = N_{\curlywedge}(\epsilon_i, k_i) N_{\curlywedge}(\epsilon_i, k_i) \, .$$

But already very nontrivial: graviton 3-vertex is already much more complicated than gravity 3 point amplitude.

Can we extend the ideas to curved space backgrounds?

- How do we define momentum eigenstates?
- What are momenta and polarization vectors?
- How can we compare them between Yang-Mills and Gravity?

# Sandwich plane waves

The Brinkman form in $d$-dimensions of the metric is

$$ds^2 = dudv - Hdu^2 - dx_a dx^a \,, \quad a = 1, .., d-2 \,.$$

with $H = H(u)_{ab} x^a x^b$, $H_a^a := H_{ab}\delta^{ab} = 0$ for vacuum.

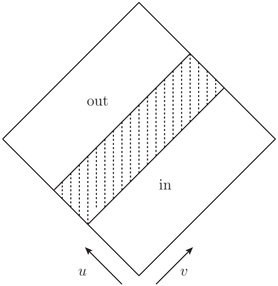


Figure: The sandwich plane wave with $x^a$-directions suppressed, $H_{ab}(u) \neq 0$ only in the shaded region with flat in- and out-regions.

- $H_{ab}$ = curvature, supported for $u \in [0, 1]$ (shaded).
- These coordinates are global, but:
- Space-time *not* globally hyperbolic! (Penrose).

# Plane wave symmetries

$$ds^2 = dudv - H_{ab}(u)x^a x^b du^2 - dx_a dx^a$$

- Symmetry group = $2d-3$-dimensional Heisenberg group transitive on $u = const.$, centre $\partial_v$.
- $e^a(u)$ with
$$\ddot{e}_a = H_{ab}e^b\,, \qquad \cdot = \frac{d}{du}$$
gives Killing vectors $e^a\partial_{x^a} - \dot{e}^a x^a \partial_v$.
- Choose $d-2$-dimensional abelian subgroup
$$D_i = e_i^a\partial_{x^a} - \dot{e}_i^a x_a \partial_v\,, \qquad i = 1 \ldots d-2\,,$$
commuting $\Leftrightarrow \dot{e}^a_{[i} e_{j]} a = 0$.
- Let $e^i_a$ be inverse matrix, $e^i_a e_{ib} = \delta_{ab}$.

# Momentum eigenstates on plane waves: I. Gravity

Friedlander showed clean cut propagation for wave equation on plane waves.

- Choose $d-1$ commuting symmetries $(\partial_v, D_i) \rightsquigarrow$ separable soln to Hamilton-Jacobi equ, momenta $(k_+, k_i)$

$$\phi_k = k_+(v + \frac{\sigma_{ab}}{2} x^a x^b) + k_i e^i_a x^a + \frac{k_i k_j F^{ij}(u)}{2k_+},$$

  where $F^{ij}(u) = \int e^i_a(u) e^{ja} du$ and $\sigma_{ab} = \dot{e}^i_a e_{bi}$.
- Then $\Phi_k = \det(e^a_i)^{-1/2} e^{i\phi_k}$ solves $\Box \Phi_k = 0$.
- For such a field define 'curved' momentum

$$K_\mu dX^\mu := d\phi_k = k_+ dv + (\sigma_{ab} x^b + k_i e^i_a) dx^a + (\ldots) du$$

**Memory:** Set $e^a_i = \delta^a_i$ as $u \to -\infty$ so $K_\mu = (k_+, k_a, k_a k^a / 2k_+)$. As $u \to +\infty$, $e^a_i(u) = b^a_i + u c^a_i$ and $\sigma_{ab} \neq 0$ so wave fronts $\phi_k = \mathit{const}$. are now curved.

# Higher spins

- We have $d-2$ covariantly constant spin raising operators

$$R^a = du\delta^{ab}\partial_{x^b} + dx^a\partial_v\,, \qquad \nabla_\mu R^a = 0\,.$$

- Gives linear gauge field on background

$$A = \frac{\epsilon_a R^a}{k_+}\Phi_k = \varepsilon_\mu dX^\mu \Phi_k\,,$$

with curved polarization $\varepsilon_\mu$, $K^\mu\varepsilon_\mu = 0$,

$$\varepsilon_\mu dX^\mu = \epsilon_a dx^a + \epsilon^a\left(\frac{k_i e^i_a}{k_+} + \sigma_{ab}x^b\right)du$$

- Linear gravity on background

$$h_{\mu\nu}dX^\mu dX^\nu = \frac{\epsilon_a R^a(\epsilon_b R^b \Phi_k)}{k_+^2} = \left((\varepsilon\cdot dX)^2 - \frac{i}{k_+}\epsilon_a\epsilon_b\sigma^{ab}du^2\right)\Phi_k$$

Note potential obstruction to double copy.

# Tails and Huygens

## Theorem (Friedlander 1970s)

*The only space-times that admit clean cut solutions to the wave equation are conformal to plane waves (or flat space).*

- Setting $\Phi = |e|^{-1/2}\delta(\phi_k)$ obtain clean cut solution to wave equation.
- Analogous spin-1 solution is
$$a = |e|^{-1/2}\epsilon_a R^a(\phi_k \Theta(\phi_k))$$
so
$$F = da = \delta(\phi_k)|e|^{-\frac{1}{2}}\epsilon_a R^a \phi_k \wedge d\phi_k + \Theta(\phi_k)|e|^{-\frac{1}{2}}\epsilon^a \sigma^0_{ab} dx^a \wedge du$$
i.e., there is backscattering with a tail.
- Similar spin-2 solution has longer tail.

# Momentum eigenstates on plane waves: II. Yang-Mills

Use same coordinates on flat space-time with gauge potential

$$A = \dot{A}_a(u) x^a du \,, \qquad F = \dot{A}_a(u) dx^a \wedge du \,.$$

Again, take sandwich wave with $Supp(\dot{A}_a) \subset u \in [0, 1]$.
Momentum eigenstate charge $e$:

$$\Phi_k = \mathrm{e}^{i\left(k_+ v + (k_a + eA_a)x^a + \frac{f(u)}{2k_+}\right)} \,, \qquad \Box_{eA}\Phi_k = 0 \,,$$

momentum $K_\mu(u) = \left(k_+, k_a + eA_a(u), \frac{\dot{f}(u)}{2k_+}\right)$ with

$$K \cdot K = 0 \quad \rightsquigarrow \quad f(u) = \int_{-\infty}^{u} (k_a + eA_a)(k^a + eA^a) du' \,.$$

**Memory:**
Choose $A_a = 0$ for $u < 0$, then for $u > 1$, $A_a(u) = const. \neq 0$.

$$K_\mu(u) = \begin{cases} \left(k_+, k_a, \frac{k_a k^a}{2k_+}\right) \,, & u < 0 \,, \\ \left(k_+, k_a + eA_a(1), \frac{\dot{f}(1)}{2k_+}\right) \,, & u > 1 \,. \end{cases}$$

# Linear YM fields on the background

Let $\mathfrak{h}$ be Cartan subalgebra of Lie algebra $\mathfrak{g}$ of gauge group $G$.

- Let $\dot{A}_a(u)x^a du$ take values in $\mathfrak{h}$.
- Encode colour $t^a$ in charge $e =$eigenvalue of $\mathfrak{h}\times$ coupling.
- Charged linear YM field $a_\mu$ satisfies

$$D^\mu D_{[\mu}a_{\nu]} + a^\mu \partial_{[\mu} eA_{\nu]} = 0\,, \qquad D_\mu = \partial_\mu + eA_\mu .$$

- Solution $a = \tilde{\epsilon}_a R^a \Phi_k = \tilde{\varepsilon}_\mu dX^\mu \Phi_k$, transverse polarization

$$\tilde{\varepsilon}_\mu(u) dX^\mu = \tilde{\epsilon}_a \left( dx^a + \frac{1}{k_+}(k^a + eA^a(u))du \right)\,, \quad \epsilon_a = const..$$

- Convention: YM background polarization vectors are tilded.

# No particle creation or leakage

As $u \to -\infty$ take linear fields to become flat space-time momentum eigenstates, i.e., $e^a_i = \delta^a_i$, and $A_a = 0$;

- $\pm$ frequency determined by sign of $k_+$, doesnt change with $u$ so no particle creation.
- Inner products are $u$-independent on both backgrounds:

  $$\langle \Phi_k | \Phi_{k'} \rangle = \int_{u=const.} \bar{\Phi}_k{}^* d\Phi_l - \Phi_l{}^* d\bar{\Phi}_k = 2k_+ \delta(k_+ - l_+)\delta^{d-2}(k_i - l_i)\,.$$

  Similarly for spin-1

  $$\langle a_1 | a_2 \rangle = \int_{u=const.} \bar{a}_1 \wedge * da_2 - a_2 \wedge {}^* d\bar{a}_1 = 2\epsilon_1 \cdot \epsilon_2 k_+ \delta(k_+ - l_+)\delta^{d-2}(k_i - l_i$$

  and spin-2

  $$\langle h_1 | h_2 \rangle = 2(\epsilon_1 \cdot \epsilon_2)^2 k_+ \delta(k_+ - l_+)\delta^{d-2}(k_i - l_i)\,.$$

- Failure of global hyperbolicity does not lead to leakage: failure is in too high co-dimension in space of null geodesics (i.e., those parallel to $\partial_v$).

# Three particle gravity amplitude

Insert linear fields into 3-vertex taken from action. In our gauge this reduces to

$$\mathcal{M}_3 = \frac{\kappa}{4} \int d^d X \left( h_1^{\mu\nu} \partial_\mu h_{2\rho\sigma} \partial_\nu h_3^{\rho\sigma} - 2 h_1^{\rho\nu} \partial_\mu h_{2\rho\sigma} \partial_\nu h_3^{\mu\sigma} \right) + perms$$

This gives with our states, after some manipulation

$$\frac{\kappa}{2} \delta^{d-1} \left( \sum_{r=1}^{3} k_r \right) \int \frac{du}{\sqrt{\det e_i^a}} \exp \left( \sum_{r=1}^{s} \frac{F^{ij} k_{ri} k_{rj}}{2 k_{r0}} \right)$$
$$\left[ \left( \varepsilon_1 \cdot \varepsilon_2 \, (K_1 - K - 2) \cdot \varepsilon_3 + \circlearrowleft \right)^2 - i k_{1+} k_{2+} k_{3+} \sigma^{ab} \mathcal{C}_a \mathcal{C}_b \right]$$

where

$$\mathcal{C}_a := \varepsilon_1 \cdot \varepsilon_2 \frac{\epsilon_{3a}}{k_{3+}} + \circlearrowleft$$

The first term is square of YM 3-pt amplitude on gravity plane wave, so tail term $\sigma^{ab} \mathcal{C}_a \mathcal{C}_b$ seems to obstruct double copy.

# Three-point YM amplitude

Three point vertex on background now obtained from cubic part of action $\int_M a_{[\mu} a_{\nu]} D^\mu a^\nu d^4 X$ which gives

$$\mathcal{A}_3 = \int du(\tilde{\varepsilon}_1 \cdot \tilde{\varepsilon}_2 \, \tilde{\varepsilon}_3 \cdot (K_1 - K_2) + \circlearrowleft) \exp\left(i \sum_{r=1}^{3} \frac{f_r(u)}{2k_{r0}}\right) C_\curlywedge(t_i)$$

in terms of flat quantities, the integrand is

$$\tilde{\epsilon}_1 \cdot \tilde{\epsilon}_2 \, \tilde{\epsilon}_3^a \left( \left( \frac{k_{1+}}{k_{2+}} k_{2a} - k_{1a} \right) + A_a \left( e_1 - e_2 \frac{k_{1+}}{k_{2+}} \right) \right) + \circlearrowleft =: F + C$$

with the second term providing background ‘tail’ dependence on $A_a$.

# Double copy replacement principle

We can establish correspondence between integrands with following rules

1. Flip charges $e_r \to -e_r$ in $\mathcal{A}_3 = F + C$ to obtain conjugate $\tilde{\mathcal{A}} = F - C$ so

$$|\mathcal{A}_3|^2 := \mathcal{A}_3 \tilde{\mathcal{A}}_3 = F^2 - C^2$$

   with $F = F(k_r, \tilde{\epsilon})$ and $C = Ck_r, \tilde{\epsilon}_r, A)$.
2. In $F$ and $C$ replace $k_{ra}$ by $k_i e^i_a$ and drop ˜s on $\epsilon$s yielding $F(k_{ri} e^i_a, \epsilon_r)$ and $C(k_{ri} e^i_a, \epsilon_r, A)$.
3. 

$$\text{Replace} \qquad e_r e_s A^a A^b \to \begin{cases} i k_{r0} \sigma^{ab} & r = s, \\ i(k_{r0} + k_{s0}) \sigma^{ab} & r \neq s\,. \end{cases}$$

This now yields a double copy relationship between the integrands that incorporates tail terms.

# Four point amplitudes: YM

Work in progress w/Adamo, Casali, Nekovar

- Construct scalar Feynman propagator as

$$G^F(X,X') = \int \frac{d^d k}{k^2 + i\epsilon} \exp i(\tilde{\phi}_k(X) - \tilde{\phi}_k(X'))$$

where $\tilde{\phi}_k$ is solution to massive Hamilton-Jacobi equation

$$\tilde{\phi}_k = k_+ v + (\mathbf{k}_a + eA_a)x^a + \frac{1}{2k_+}\int^u ds[k^2 + (\mathbf{k} + eA(s))^2]$$

where now $k^2 = k_+k_- - k_a k^a \neq 0$.

- For spin-1 must invert $(\Box_{eA} + k^2)a_\mu + 2ieF_\mu^\nu a_\nu = 0$ to get

$$G^F_{\mu\nu} = \int \frac{d^d k}{k^2 + i\epsilon} P_{\mu\nu}(u,u',k_+) \mathrm{e}^{i(\phi_k(X) - \phi_k(X'))}$$

$$P_{\mu\nu} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -\delta_{ac} & \alpha\Delta A_a \\ 1 & -\alpha\Delta A_c & \frac{\alpha^2}{2}\Delta A^2 \end{pmatrix}, \quad \Delta A = A(u) - A(u'), \quad \alpha = \frac{ie}{k_+}$$

Gravity similar, but *much* more complicated.

# Double copy at four points

YM diagrams have two 3-vertices and propagator or one 4-vertex.

- To find BCJ form must open up 4-vertex into 3-vertices.
- Then 3-channels, $s$, $t$ and $u$ in flat space give

$$\mathcal{A}_4 = \frac{N_s C_s}{s} + \frac{N_t C_t}{t} + \frac{N_u C_u}{u}$$

  with $s = (k_1 + k_2)^2$, $t = (k_2 + k_3)^2$, $u = (k_1 + k_3)^2$.
- Jacobi-identity is $C_s + C_t + C_u = 0$: for BCJ property need

$$N_s + N_t + N_u = 0\,,$$

  always true in flat space; is it true on a plane wave?
- Do propagators double copy? Perhaps need convolution.

# Further remarks and developments

We have seen that explicit Feynman rules on plane wave can be set up for both gravity and YM.
These satisfy some double copy rules at 3pts but work is in progress at four points.

- Original motivation was to provide independent calculation to compare to ambitwistor string computation on plane wave background (which expresses double copy): we obtained agreement with these results in arxiv:1708.09249.
- The original double copy is multiplication in momentum space, and so is convolution on space-time in general.
- For Feynman propagators on backgrounds for different spins convolution must play bigger role.
- Full geometric understanding of double copy in nonlinear regime is a long way off, but framework is useful now for gravitational wave signatures etc..

# The end

Thank You!