Logarithmic decay of waves on spacetimes bounded by Killing horizons

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Overview

- We study the wave equation on spacetimes bounded by Killing horizons modeling the event horizons of your favorite non-rotating black holes.
- Without any assumptions on the behavior of null-geodesics, solutions of the wave equation exhibit logarithmic energy decay up to the horizon.
- The resolvent grows at most exponentially with frequency.

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- The resolvent grows at most exponentially with frequency.

Results of this kind are well known in other geometric settings (to be discussed)

Geometric setup

Let $(M = \mathbb{R}_t \times X, g)$ be a stationary Lorentzian spacetime with compact time slices. This means:

- $X \simeq \{t = 0\}$ is compact, connected, and spacelike,
- ∂_t is a Killing vector field.

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We impose two additional hypotheses:

- $\partial M \neq \emptyset$ is a connected Killing horizon generated by ∂_t with positive surface gravity,
- ∂_t is timelike in the interior: $g(\partial_t, \partial_t) > 0$ on M°.

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- There is a function $\kappa : \mathcal{H} \to \mathbb{R}$, called the surface gravity, such that on $\mathcal{H}$,

$$\nabla_g(g(T, T)) = -2\kappa T.$$

- κ is constant along integral curves of T.

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If $(r, y) \in [0, \varepsilon)_r \times \partial X_y$ are normal coordinates on $X = \{t = 0\}$,

$$g^{-1} = -2\kappa r \partial_r^2 - 2\partial_t \partial_r - k(y, \partial_y) + \partial_t^2 + \mathrm{l.o.t}$$

near ∂M, where k is a (dual) Riemannian metric on ∂X.

Examples

Schwarzschild metric in Eddington–Finkelstein coordinates:

$$g = (1 - 2m/r)dt_*^2 - 2dt_* dr - r^2 d\mathbb{S}^2$$

on $\mathbb{R}_{t_*} \times [2m, \infty) \times \mathbb{S}^2$ (but time slices are not compact).

The event horizon $\{r = 2m\}$ is a Killing horizon generated by ∂_{t_*}, with $\kappa = (4m)^{-1}$.

A compact example: stationary perturbations of Schwarzschild-de Sitter spacetime preserving both horizons.

Energy decay

Since X is spacelike, the energy (measured by stress-energy tensor)

$$E[v](s) = \int_{\{t=s\}} |Nv|^2 - (1/2)g^{-1}(dv, d\bar{v})\, dS_X$$

associated with the timelike unit normal N to $\{t = s\}$ controls all first order derivatives.

Redshift effect: if $\kappa > 0$, then for each solution of $\Box_g v = 0$,

$$E[v](t) \lesssim E[v](0).$$

Does $E[v](t)$ decay, and if so, how rapidly?

Energy decay

In general, the answer depends on trapping. A null-geodesic $\gamma(s)$ with $\gamma(0) \in M^\circ$ is non-trapped if $\gamma(s) \to \mathcal{H}$ as $s \to \pm\infty$.

If M is non-trapping and $\Box_g v = 0$ with initial data $(v_0, v_1) \in H^2(X) \times H^1(X)$, then for some $\nu > 0$,

$$E[v](t) \lesssim e^{-\nu t} \|(v_0, v_1)\|_{H^2 \times H^1}.$$

This also holds when trapping is sufficiently mild.

Energy decay

When trapping is strong, there exist examples where

$$E[v](t) \lesssim \frac{p(t)}{\log(2+t)} \|(v_0, v_1)\|_{H^2 \times H^1}$$

is false for any function $p(t)$ tending to zero.

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How about upper bounds without any assumptions on trapping?

Theorem (G' 17)

$E[v](t)$ decays logarithmically in time with a loss of derivatives,

$$E[v](t)^{1/2} \lesssim (\log(2+t))^{-1} \|(v_0, v_1)\|_{H^2 \times H^1}$$

provided $\Box_g v = 0$ with initial data $(v_0, v_1) \in H^2(X) \times H^1(X)$.

Decay to a constant

Solutions decay in norm to a constant: given (v_0, v_1), define

$$v_\infty = \mathrm{vol}(\partial X)^{-1} \int_X v_1 - (2W - \mathrm{div}_g W) v_0 \, dS_X,$$

where W is orthogonal projection of ∂_t onto TX.

Theorem (G' 17)

With $(v_0, v_1) \in H^2(X) \times H^1(X)$ *and* $v_\infty \in \mathbb{C}$ *as above,*

$$\|v(t) - v_\infty\|_{H^1(X)} + \|\partial_t v(t)\|_{L^2(X)} \lesssim (\log(2+t))^{-1} \|(v_0, v_1)\|_{H^2 \times H^1}$$

provided $\Box_g v = 0$ *with initial data* (v_0, v_1).

The same result holds for higher Sobolev norms, giving pointwise logarithmic decay for smooth initial data.

There are many works on unconditional logarithmic energy decay for different operators and geometries:

Burq '98, '02, Vodev '00, Cardoso–Vodev '02, Bellassoued '03, Fu '08, Fathallah '09, Bouclet '10 Rodnianski–Tao '11, Eller–Toundykov '12, Holzegel–Smulevici '13, Datchev '14, Burq–Joly '14, Moschidis '15, Buffe '16, Hassine '15, Cornilleau '14, Shapiro '16, '17

Moschidis: Lorentzian spacetimes containing more general horizons, but also at least one asymptotically flat end.

Geodesic flow near Killing horizons

Let (t, x) be coordinates on $M = \mathbb{R}_t \times X$ and (τ, ξ) be dual momenta, i.e. write covectors as

$$\tau\, dt + \xi \cdot dx.$$

Let $G : T^*M \to \mathbb{R}$ be the principal symbol of $\Box_g$, so that

$$G(x, \xi, \tau) = -|\tau\, dt + \xi \cdot dx|_g^2$$

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Recall ∂_t is timelike in $M^\circ \implies G$ is positive definite on $T^*M^\circ \cap \{\tau = 0\}$, i.e.

$$G(x, \xi, 0) = -\big|\xi \cdot dx\big|_g^2 > 0 \text{ for } \xi \neq 0.$$

Geodesic flow near Killing horizons

The Hamilton vector field H_G generates the null geodesic flow on $\{G = 0\} \setminus 0$:

$$H_G = (\partial_\tau G)\partial_t - (\partial_t G)\partial_\tau + (\partial_{\xi_i} G)\partial_{x^i} - (\partial_{x^i} G)\partial_{\xi_i}$$

What happens over the boundary? $\mathcal{H}$ is characteristic for $\Box_g$, and in fact

$$\{G = 0, \tau = 0\} \setminus 0 = N^*\partial M \setminus 0.$$

Thus $N^*\partial M \setminus 0$ is invariant under the flow. On the other hand,

$$\pm H_G r = \pm 2\tau > 0 \text{ on } \{r = 0,\ G = 0,\ \pm\tau > 0\}.$$

Pseudoconvexity

A hypersurface $\{\phi = 0\}$ is pseudoconvex w.r.t $\{\phi > 0\}$ if

$$\phi(t,x) = G(x,\xi,\tau) = H_G\phi(x,\xi,\tau) = 0$$
$$\implies H_G^2\phi(x,\xi,\tau) > 0.$$

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If $r_0 > 0$ is sufficiently small, then $\{r = r_0\}$ is pseudoconvex with respect to $\{r < r_0\}$,

∂M is certainly not pseudoconvex because of trapped null-bicharacteristics when $\tau = 0$ (but recall $H_G r \neq 0$ when $\tau \neq 0$).

Resolvent

Define the stationary wave operator $P(\omega) = e^{-i\omega t}\Box_g e^{i\omega t}$ on X.

Kernel of $P(\omega)$ corresponds to a mode solutions $v(t,x) = e^{i\omega t}u(x)$ of $\Box_g v = 0$.

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Theorem (Mazzeo-Melrose '87, Vasy '10, Warnick '13)

The operator $P(\omega) : H^1(X) \to L^2(X)$ is Fredholm of index zero in the half-plane $\{\mathrm{Im}\,\omega < \kappa/2\}$, and $P(\omega)^{-1}$ is meromorphic.

Poles of $P(\omega)^{-1}$ are called resonances or quasinormal modes, forming a discrete subset of $\{\mathrm{Im}\,\omega < \kappa/2\}$.

Energy decay

Theorem (Lebeau '96, Burq '98, Batty–Duyckaerts '08)

To prove logarithmic energy decay, it suffices to prove a high-energy resolvent estimate *for $|\omega| \gg 1$:*

$$\|P(\omega)^{-1} f\|_{H^1(X)} \lesssim e^{C|\omega|} \|f\|_{L^2(X)}.$$

Part of this theorem requires ruling out resonances on the real axis.

- For $\omega \neq 0$ this is a version of Rellich's theorem.
- There is a resonance at $\omega = 0$ generated by constants, but $E[v](t)$ does not see this.

For decay to a constant, need to analyze zero resonance.

Resolvent bounds

First step is to prove a Carleman estimate. Define the conjugated operator

$$P_\varphi(\omega) = e^{\omega\varphi} P(\omega) e^{-\omega\varphi},$$

where $\omega > 0$ plays the role of a large parameter.

Theorem (G '17)

There exists $\varphi \in \mathcal{C}^\infty(X)$ and such that if $\omega \gg 1$, then

$$\omega^{3/2}\|u\|_{L^2(X)} + \omega^{1/2}\|u\|_{H^1_b(X)} \lesssim \|P_\varphi(\omega)u\|_{L^2(X)} + \omega^{3/2}\|u\|_{L^2(\partial X)}$$

Resolvent bounds

Applying this to $e^{|\omega|\varphi}u$ yields for $\omega \gg 1$,

$$\|u\|_{H^1_b(X)} \lesssim e^{C|\omega|}\left(\|P(\omega)u\|_{L^2(X)} + \|u\|_{L^2(\partial X)}\right)$$

Two problems to be resolved:

- H^1_b norm only controls first order derivatives tangent to ∂X,
- There is an extra boundary term (in L^2 norm) on the right.

Carleman estimates in the interior

Recall $P(\omega)$ is an operator on X depending on $\omega \in \mathbb{C}$.

The semiclassical (or parameter-dependent) symbol of $P_\varphi(\omega)$ is

$$\begin{aligned} G_\varphi(x,\xi,\omega) &= G(x,\xi + i|\omega| d\varphi, \omega) \\ &= -\Big|\omega\, dt + (\xi \cdot dx + i|\omega|\, d\varphi)\Big|_g^2 \end{aligned}$$

where $(x,\xi) \in T^*X$. Thus $G_\varphi(x,\xi,\omega)$ is homogeneous in (ξ,ω).

Since ∂_t is timelike in M°, have that $G_\varphi(x,\xi,0) = G(x,\xi,0)$ is positive definite on T^*X°

Carleman estimates in the interior

First consider a compact set $K \Subset X^\circ$ and $v \in \mathcal{C}^\infty_{\mathrm{c}}(K)$. Integrating by parts,

$$\begin{aligned}\|P_\varphi(\omega)u\|^2_{L^2(X)} &= \langle P_\varphi(\omega)P_\varphi(\omega)^*u, u\rangle \\ &\quad + \langle[\mathrm{Re}\, P_\varphi(\omega), \mathrm{Im}\, P_\varphi(\omega)]u, u\rangle\,.\end{aligned}$$

Suffices to construct $\varphi \in \mathcal{C}^\infty(X)$ such that over K,

$$|G_\varphi|^2 + \{\mathrm{Re}\, G_\varphi, \mathrm{Im}\, G_\varphi\} \gtrsim \omega(|\xi|^2 + \omega^2).$$

By homogeneity in (ξ, ω), it suffices to satisfy the bracket condition

$$\omega^{-1}\{\mathrm{Re}\, G_\varphi, \mathrm{Im}\, G_\varphi\}(x, \xi, \omega) > 0$$

when $G_\varphi(x, \xi, \omega) = 0$ and $\xi^2 + \omega^2 = 1$.

Carleman estimates in the interior

As $\omega \to 0$, the bracket condition reads

$$H_G^2 \varphi(x, \xi, 0) > 0 \text{ when } G(x, \xi, 0) = 0.$$

But recall $G(x, \xi, 0)$ is positive definite, so bracket condition when $\omega = 0$ is trivial.

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For $\omega \neq 0$ and $K \Subset X^\circ$ we can set $\varphi = e^{\alpha\psi}$ for $\alpha \gg 1$ depending on K, where $d\psi \neq 0$.

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For $\omega \neq 0$ and $K \Subset X^\circ$ we can set $\varphi = e^{\alpha\psi}$ for $\alpha \gg 1$ depending on K, where $d\psi \neq 0$.

This implies the Carleman estimate for $u \in \mathcal{C}_c^\infty(K)$ and $\omega \gg 1$,

$$\omega^{3/2} \|u\|_{L^2(X)} + \omega^{1/2} \|u\|_{H^1(X)} \lesssim \|P_\varphi(\omega) u\|_{L^2(X)},$$

Carleman estimates up to the boundary?

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Carleman estimates up to the boundary?

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If $\varphi = \varphi(r)$ and $\varphi'(r) < 0$, then φ automatically satisfies the bracket condition over ∂X for $\omega \neq 0$, since

$$(H_G r)(x, \xi, \omega) \neq 0 \text{ when } r = 0,\ G(x, \xi, \omega) = 0,\ \omega \neq 0.$$

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Unfortunately, when $\omega = 0$ the bracket condition is not satisfied since this would imply pseudoconvexity of ∂M!

A degenerate Carleman estimate

Integrating by parts up to ∂M in spacetime resolves some of these problems.

Suppose that $\varphi'(r) < 0$ and for $\lambda(r) \in \mathcal{C}^\infty$ to be chosen, set

$$\Pi(x, \xi, \tau) = -|\tau|\varphi'(r)\{G, r\}^2 - \{G, \{G, r\}\} + 4\lambda G.$$

Is Π positive definite in (ξ, τ) up to ∂X?

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Is Π positive definite in (ξ, τ) up to ∂X?

No, because this again would imply pseudoconvexity.

On the other hand,

$$C_0\{G, r\}^2 - \{G, \{G, r\}\} + 4\lambda G \gtrsim ((r\rho)^2 + |\eta|^2 + \tau^2).$$

for $C_0 \gg 1$ and $\lambda(r) = -\kappa + (1-\delta)rC_0$.

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for $C_0 \gg 1$ and $\lambda(r) = -\kappa + (1-\delta)rC_0$.

Let $v(t,x) = e^{i\omega t}u(x)$ with $\omega > 0$.

Thus $\Pi(dv, d\bar{v})$ controls $|r\partial_r u|^2 + |\partial_y u|^2 + \omega^2|u|^2$ near ∂M for $\omega \gg 1$, so

$$\int_X \Pi(dv, d\bar{v})\, dS_X \gtrsim \|u\|^2_{H^1_b(X)} + \omega^2\|u\|^2_{L^2(X)}.$$

A degenerate Carleman estimate

Let $\Box_\varphi = e^{\omega\varphi}\Box_g e^{-\omega\varphi}$, where $\varphi = \varphi(r)$ and $\varphi'(r) < 0$.

Integrate by parts to obtain an inequality

$$\int_X \omega^{-1}|\Box_\varphi v|^2 \, dS_X + \int_{\partial X} |\partial_t v|^2 \gtrsim \int_X \Pi(dv, d\bar{v}) + \omega^2 V|v|^2 \, dS_X$$

for $v(t,x) = e^{i\omega t}u(x)$ with $\omega > 0$, supported near ∂M. Here, and $V = V(r)$ is a potential term.

Unfortunately, $V = -r(\varphi')^2 - 2r^2\varphi'\varphi''$. Thus the most subtle part of the argument is constructing φ, with

$$\varphi = \varphi(r) \text{ and } \varphi'(r) < 0 \text{ near } \partial X,$$

satisfying the bracket condition everywhere on T^*X°.

Once φ is fixed, $\omega^2 V$ can be absorbed if u is supported sufficiently close to ∂X.

A degenerate Carleman estimate

Similarly, the boundary integral contributes $\omega^2 \|u\|^2_{L^2(\partial X)}$, so

$$\omega^3 \|u\|_{L^2(\partial X)} + \|P_\varphi(\omega)u\|^2_{L^2(X)} \gtrsim \omega \|u\|^2_{H^1_b(X)} + \omega^3 \|u\|^2_{L^2(X)}$$

for u supported near ∂X.

Combine with the interior estimates via cutoffs, completing the proof of the Theorem.

Eliminating the boundary term

Apply spacetime Green's formula to $v = e^{i\omega t} u$ with $\omega \in \mathbb{R}$.

Since ∂_t is normal to ∂M,

$$\mathrm{Im} \int_{\partial X} \bar{v}\, \partial_t v \, dS_{\partial X} = \mathrm{Im} \int_X \bar{v}\, \Box_g v \, dS_X .$$

Apply Cauchy–Schwarz to estimate

$$e^{C\omega} \|u\|_{L^2(\partial X)} \leq C_\varepsilon e^{C'\omega} \|P(\omega)\|_{L^2(X)} + \varepsilon \|u\|_{L^2(X)}$$

and absorb the second term on RHS. Therefore,

$$\|u\|_{H^1_b(X)} \leq e^{C\omega} \|P(\omega) u\|_{L^2(X)} .$$

Improving the weight

Upgrading H^1_b to H^1 norm again uses the positivity of κ via

$$\|u\|_{H^1(X)} \lesssim \|P(\omega)u\|_{L^2(X)} + (1+\omega)\|u\|_{L^2(X)}.$$

The proof is exactly the same redshift argument that establishes energy boundedness statement or Fredholmness of $P(\omega)^{-1}$.

This proves the resolvent estimate, since $P(\omega)$ is Fredholm of index zero and we have shown that for $\omega \gg 1$,

$$\|u\|_{H^1(X)} \lesssim e^{C\omega}\|P(\omega)u\|_{L^2(X)}.$$

Open questions

What can be said about rotating spacetimes? Here, ∂_t is not timelike in an ergoregion $\Longrightarrow$ superradiance phenomena.

- Dold '16: there is a rotating spacetime (with additional timelike boundary) admitting resonances in lower half-plane (cf. Shlapentokh–Rothman, Moschidis)
- Under an priori energy boundedness assumption are there conditions under which you can exclude nonzero real resonances and prove resolvent bounds?