The Reeh-Schlieder Property and its Relevance for Scattering Theory

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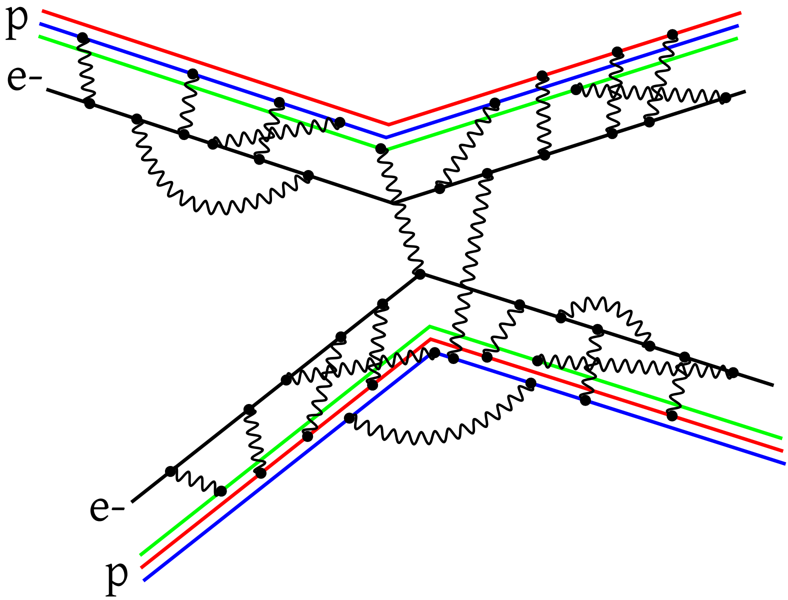
Quantum Fields, Scattering, and Space-Time Horizons: Mathematical Challenges,
Les Houches, May 23rd 2018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Interacting Quantum Field Theory, Non-Perturbatively

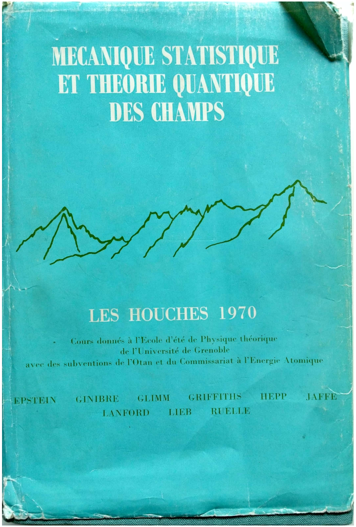
Exercise 1 Quantum Mechanics:

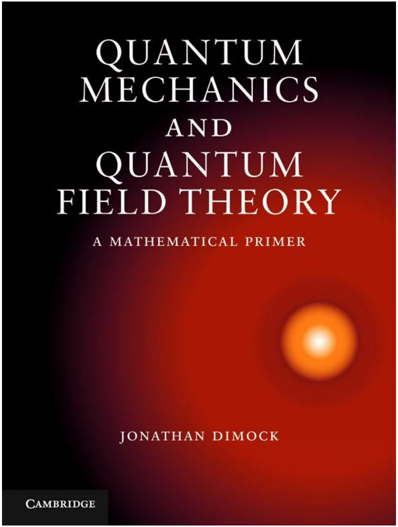
(a) Find $\mathscr{H}$, Hamiltonian H_0 and Observables for free particles
(b) Born probability interpretation $|\Psi(x)|^2$
(c) Add interaction $H := H_0 + H_{\text{int}}$

Exercise 2 Constructive Quantum Field Theory:

(a) Discover Free Quantum Fields $\phi_0(x)$, $\mathscr{H}_0$, H_0
(b) Interpretation of $(\phi_0, \mathscr{H}_0, H_0)$ in terms of free particles
(c) ϕ_0 implements Einstein-Causality quantum mechanically

From now on may assume for simplicity spacetime-dim. $1+1$
(e) add Interaction $H^R_{\text{int}} = \int_{|x|<R} \mathrm{d}x \; :\phi^4(x):$, goal $R \to \infty$
(f) construct local algebras ($R \to \infty$, via hyperbolicity "$c < \infty$")
(g) $H^R = H_0 + H^R_{\text{int}}$ has ground state $\Omega^R \xrightarrow{R\to\infty} 0$.
(h) Still $\omega(A) := \lim_{R\to\infty} \langle \Omega^R, A\Omega^R \rangle$ yields a well-defined state, if restricted to the algebra of **local observables**.
(j) ω defines new Hilbert space $\mathscr{H}$ on which interact. model lives (change of rep.), and where $H = \lim\limits_{R\to\infty} H^R$ is well-defined.


MECANIQUE STATISTIQUE
ET THEORIE QUANTIQUE
DES CHAMPS
LES HOUCHES 1970
Cours donnés à l'Ecole d'été de Physique théorique
de l'Université de Grenoble
avec des subventions de l'Otan et du Commissariat à l'Energie Atomique
EPSTEIN GINIBRE GLIMM GRIFFITHS HEPP JAFFE
LANFORD LIEB RUELLE


QUANTUM
MECHANICS
AND
QUANTUM
FIELD THEORY
A MATHEMATICAL PRIMER
JONATHAN DIMOCK
CAMBRIDGE

Overview

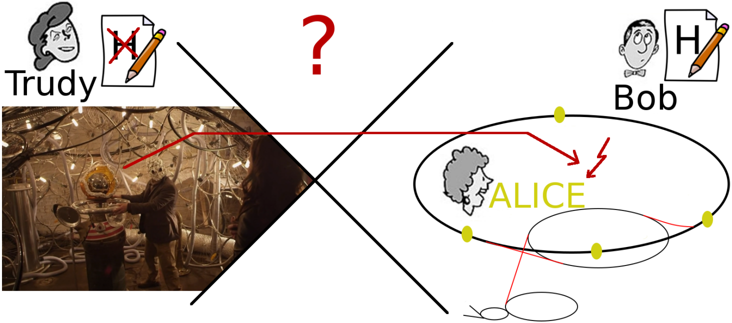
Non-Locality of the Vacuum: Reeh-Schlieder Property

Local Observables $A \in \mathfrak{A}(\mathcal{O}) \subset B(\mathscr{H}) \sim$ bounded functions of Fields $\phi(x)$ smeared with test functions compactly supported in $\mathcal{O}$.

Cyclicity of the vacuum Ω: $\overline{\mathfrak{A}\Omega} = \mathscr{H}$ for $\mathfrak{A} := \bigcup_{\mathcal{O}} \mathfrak{A}(\mathcal{O})$ (HK6$^\flat$)

Reeh-Schlieder (1961): $\overline{\mathfrak{A}(\mathcal{O})\Omega} = \mathscr{H}$ for any open $\mathcal{O} \neq \emptyset$ (HK6)

Rem.: (HK6$^\flat$) + "Additivity" $\mathfrak{A} \subset \left(\bigvee_x \mathfrak{A}(\mathcal{O}_0 + x) \right)''$ $\Longrightarrow$ (HK6)



Standard Proof strategy for Reeh-Schlieder

- Positivity of Energy, and Locality imply certain Analyticity!
- **Free Scalar QF:** $(\phi_0(f) + \pi_0(g))\Omega = \omega_m^{-1/2}\tilde{f} + \omega_m^{1/2}\tilde{g}$, $f, g \in \mathscr{S}(\mathbb{R}^s)$. Then **Anti-locality** of $\omega_m := (-\Delta^2 + m^2)^{\frac{1}{2}}$,
$$T\big|_{\mathcal{O}} = 0, \;\; \omega_m T\big|_{\mathcal{O}} = 0 \implies T = 0.$$
$(m > 0)$, $T \in \mathscr{S}'(\mathbb{R}^s)$, region $\mathcal{O} \subset \mathbb{R}^s$. (Segal, Goodman'65)
- **General Argument (sketch):** $\Psi \in (\mathfrak{A}(\mathcal{O} + B_\epsilon)\Omega)^\perp$ $(\epsilon > 0)$,
$$0 = \langle \Psi, U(t, \mathbf{x}) A\Omega \rangle \quad \forall\, (t, \mathbf{x}) \in B_\epsilon(0)$$
boundary value of $f(t, \mathbf{x}) := \langle \Psi, \mathrm{e}^{\mathrm{i}tH - \mathrm{i}\mathbf{x}\cdot\mathbf{P}} A\Omega \rangle$, $(t, \mathbf{x}) \in \mathbb{C}^{s+1}$, f holomorphic on $\{(t, \mathbf{x}) : \mathrm{Im}(t\omega - \mathbf{x}\cdot\mathbf{k}) > 0, (\omega, k) \in \bar{V}^+\}$.
$$\overset{\text{(EotW Thm.)}}{\Longrightarrow} f(t, \mathbf{x}) = 0 \;\forall (t, \mathbf{x}) \Longrightarrow \Psi \in \Big(\bigcup_{x \in \mathbb{R}^{s+1}} \mathfrak{A}(\mathcal{O} + x)\Omega\Big)^\perp.$$

Pathology of Minkowski background?
No! $\longrightarrow$ Strohmaier-Verch-Wollenberg'02, Gérard-Wrochna'17

Algebraic Framework for Local Quantum Theory

Mathematical Objects

Haag-Kastler QFT $(\mathfrak{A}, \alpha, \Omega, \mathscr{H})$ in the vacuum sector.

Described by mathematical entities...

- Hilbert space $\mathscr{H}$ of pure states
- distinguished *vacuum* $\Omega \in \mathscr{H}$
- net of von Neumann algebras $\mathbb{R}^{3+1} \supset \mathcal{O} \mapsto \mathfrak{A}(\mathcal{O}) \subset \mathrm{B}(\mathscr{H})$
- space-time translations of states $(t, \mathbf{x}) \mapsto U(t, \mathbf{x}) = e^{\mathrm{i}tH - \mathrm{i}\mathbf{x}\cdot\boldsymbol{P}}$
- translations of observables $\alpha_x A := A(x) := U(x)\, A\, U(x)^*$

Algebraic Framework for Local Quantum Theory

Haag-Kastler Axioms

. . . which are subject to

(HK1) $\mathcal{O}_1 \subset \mathcal{O}_2 \Longrightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$ **(Isotony)**

(HK2) $\mathcal{O}_1 \subset \mathcal{O}_2' \Longrightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)'$ **(Locality)**

(HK3) $\alpha_x \mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{O} + x), \ \forall x \in \mathbb{R}^4$ **(Covariance)**

(HK4) $E_{(H,\boldsymbol{P})}(\{0\})\mathscr{H} = \mathbb{C}\Omega$ (**Uniqueness of $\boldsymbol{\Omega}$**)

(HK5) $\operatorname{supp} E_{(H,\boldsymbol{P})} \subset \bar{V}^+$ (**Spectrum Condition**)

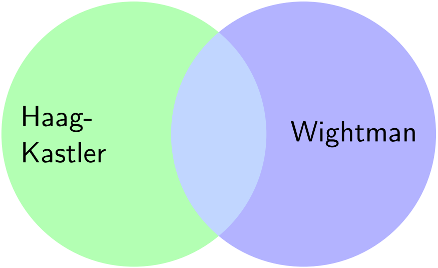
(HK6) $\overline{\mathfrak{A}(\mathcal{O})\Omega} = \mathscr{H}$ (**Reeh-Schlieder Property**)

Algebraic Framework for Local Quantum Theory

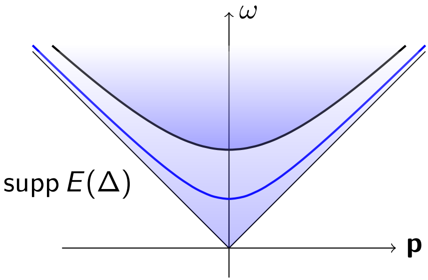
Example

Take physics-textbook scalar free field $\phi(x^\mu)$, $\mathcal{O} \subset \mathbb{R}^{s+1}$ bounded.

$$\begin{aligned}
\mathscr{H} &:= \text{symm. Fock space} \\
\mathfrak{A}(\mathcal{O}) &:= \overline{\operatorname{span}\{\mathrm{e}^{\mathrm{i}\phi(f)} : f \in \mathscr{S}(\mathbb{R}^{s+1}),\ \operatorname{supp} f \subset \mathcal{O}\}}^{\text{w.o.t.}} \\
\phi(f) &:= \int \mathrm{d}^{s+1}x\, f(x)\phi(x) \\
\alpha_x(\mathrm{e}^{\mathrm{i}\phi(f)}) &:= \mathrm{e}^{\mathrm{i}\phi(T_x f)}, \quad (T_x f)(y) := f(y-x) \\
\Omega &:= \text{Fock vacuum}
\end{aligned}$$



The Particle Spectrum



Vacuum $\Omega \in \mathscr{H}$ translation invariant, Space-time translations α_x unitarily implemented

$$\mathscr{H} \ni \Psi \longmapsto U(t, \mathbf{x})\Psi$$

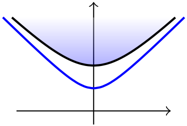
SNAG-Theorem $\rightarrow$ strongly commut. self-adjoint generators $(H, \boldsymbol{P})$
$\hat{=}$ energy-momentum op.

Spectral Resolution of $(H, \boldsymbol{P})$ by POVM $E(\Delta)$ for Borel $\Delta \subset \mathbb{R}^4$.

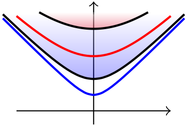
Def. (Wigner particle) Single-particle states are eigenvectors $\Psi_1 \in \mathscr{H}$ of the relativistic mass operator $M^2 = H^2 - \boldsymbol{P}^2$.

Scattering Theory: What is known rigorously?

- Haag '54, Lehmann-Symanzik-Zimmermann '54
  Postulated Asymptotic Condition:
  "Interacting $\phi(x^\mu) \xrightarrow{x^0 \to \pm\infty} \phi_0^\pm(x^\mu)$ free"

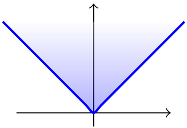


- Haag '59: establish "out"-products of 1-p. vectors
  Ruelle '62, Hepp '65 — proof, isolated mass shell

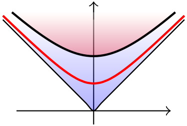


- Herbst '71 — isolated vacuum,
  "spectral condition" (SC),

  i.e. need local operator $A \in \mathfrak{A}(\mathcal{O})$ s.t. $A\Omega$ has "nicely behaved" spectrum near mass shell



- Buchholz '77 — **no (SC)** nor other conditions needed for $m = 0$ via Huygens' principle



- Dybalski '05 — (SC) + non-isolated vacuum

- Duch, Herdegen '13 — (SC) weakened, $m \geq 0$

Preparing Single-Particle States

Single-particle states $\Psi_1, \Psi_2 \in E_{\{M=m\}}\mathscr{H}$ are non-local objects:

$$\Psi_1 = E_m A\Omega = \chi(\tfrac{M^2-m^2}{\epsilon})A\Omega \sim A(\hat{\chi}_\epsilon)\Omega, \quad (\chi \in \mathscr{S}, \epsilon \searrow 0).$$

Instead now fix **one** bounded space-time region $\mathcal{O} \subset \mathbb{R}^4$.

Reeh-Schlieder (HK6) $\Rightarrow \exists (A_{k\beta})_{\beta>0} \subset \mathfrak{A}(\mathcal{O})$: $\|A_{k\beta}\Omega - \Psi_k\| = \beta$.

Def.: We call a family of local operators $(A_{k\beta})_{\beta>0} \subset \mathfrak{A}(\mathcal{O})$ s.t.

$$\|A_{k\beta}\Omega - \Psi_k\| \leq \beta \text{ and } \|A_{k\beta}\| \leq \beta^{-\gamma}$$

a Reeh-Schlieder family for Ψ_k of degree $\gamma > 0$.

Assumption: **Strengthened Reeh-Schlieder Property** (HK6$^\sharp$)

Reeh-Schlieder families of **finite** degree generate
a total subset of the single-particle space $\mathscr{H}_1 \subset \mathscr{H}$.

Strengthened Reeh-Schlieder yields Scattering States

Strengthened Reeh-Schlieder Property ($\gamma > 0$)
$(A_{k\beta})_{\beta>0} \subset \mathfrak{A}(\mathcal{O})$, s.t. $\|A_{k\beta}\Omega - \Psi_k\| \leq \beta$ and $\|A_{k\beta}\| \leq \beta^{-\gamma}$

Theorem (MD'15) Let Ψ_k be single-particle states admitting Reeh-Schlieder families $A_{k\beta}$ of finite degree. Then for any regular positive-energy Klein-Gordon sol. f_k with disjoint velocity supports

$$\Psi_\tau := \mathcal{B}_{1\tau} \ldots \mathcal{B}_{n\tau}\Omega \overset{\tau\to\pm\infty}{\longrightarrow} \Psi^\pm$$

The scalar products of any two such Ψ^+, Ψ'^+ can be computed using the Fock prescription (similarly for incoming states).

Previous results (Herbst '71, Dybalski '05, Herdegen '13) require spectral condition of Herbst-type, e.g. for some $\epsilon > 0$,

$$\Psi_k = E_{\{M=m\}}A_k\Omega, \quad A_k \in \mathfrak{A}(\mathcal{O}), \quad \left\|E_{\{0<|M-m|<\delta\}}A_k\Omega\right\| \leq \delta^\epsilon.$$

Construction of Scattering States

Reeh-Schlieder and Haag-Ruelle Creation Operators

Reference Dynamics: Klein-Gordon solutions f_k with disjointly and compactly supported wave packets $\tilde{f}_k \in \mathscr{C}_c^\infty(\mathbb{R}^3)$ ("*regular*")

Creation-Operator Approximants: with $\hat{\chi} \in \mathscr{C}_c^\infty(\mathbb{R}^4 \setminus \bar{V}^-)$, set

$$B_{k\beta} := A_{k\beta}(\chi) := \int \mathrm{d}^4 x \; \chi(x) \; A_{k\beta}(x),$$

$$\boxed{\mathcal{B}_{k\tau} := \int \mathrm{d}^3 x \; f_k(\tau, \mathbf{x}) \; B_{k\beta}(\tau, \mathbf{x}), \quad (\tau \in \mathbb{R}).}$$

Haag-Ruelle/LSZ: $\mathcal{B}_{k\tau}\Omega \rightharpoonup \Psi'_k(f_k) := \tilde{f}_k(\mathbf{P})\Psi'_k \in \mathscr{H}_1$ for fixed small enough β.

Reeh-Schlieder: $\boxed{\beta = \beta(\tau) := |\tau|^{-\mu}, \; \mu > 0}$ then $\mathcal{B}_{k\tau}\Omega \to \Psi_k(f_k)$.

Candidate Scattering States: Limits $\tau \to \pm\infty$ of $\Psi_\tau := \mathcal{B}_{1\tau}\mathcal{B}_{2\tau}\Omega$.

Mathematical Tools (1) — Discretized Cook's method

$$\|\Psi_{\tau_2} - \Psi_{\tau_1}\| = \left\| \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \, \partial_\tau \Psi_\tau \right\| \leq \int_{\tau_1}^{\tau_2} \mathrm{d}\tau \ \|\partial_\tau \Psi_\tau\| \overset{!}{<} \infty \quad (\tau_2 \to \pm\infty)$$

$$\|\Psi_{\tau_N} - \Psi_{\tau_1}\| \leq \sum_k \left\| \mathcal{B}_{1\tau_{k+1}} \mathcal{B}_{2\tau_{k+1}} \Omega - \mathcal{B}_{1\tau_k} \mathcal{B}_{2\tau_k} \Omega \right\| \overset{!}{<} \infty \quad (\tau_N \to \pm\infty)$$

$$\begin{aligned} \|\Psi_{\tau_2} - \Psi_{\tau_1}\| &\leq \|\mathcal{B}_{1\tau_2}(\mathcal{B}_{2\tau_2} - \mathcal{B}_{2\tau_1})\Omega\| + \|(\mathcal{B}_{1\tau_2} - \mathcal{B}_{1\tau_1})\mathcal{B}_{2\tau_1}\Omega\| \\ &\leq \|\mathcal{B}_{1\tau_2}(\mathcal{B}_{2\tau_2} - \mathcal{B}_{2\tau_1})\Omega\| + \|\mathcal{B}_{2\tau_1}(\mathcal{B}_{1\tau_2} - \mathcal{B}_{1\tau_1})\Omega\| \quad (\star) \\ &\qquad + (\text{commutators}) \quad (\star\star) \end{aligned}$$

Recall: $\mathcal{B}_{j\tau}\Omega \to \Psi_j \in \mathscr{H}_1$ (by construction)

For best possible **summability** as $N \to \infty$ we should

- choose $(\tau_k)_{k\in\mathbb{N}}$ as sparse as possible, $\tau_k := (1+\rho)^k \tau_0$, $\rho > 0$
- control equal- and non-equal-time commutators in $(\star\star)$
- control estimation of unbounded leftmost $\mathcal{B}_{j\tau_k}$ in $(\star)$

Tools (2) — Non-Equal-Time Commutator Estimates

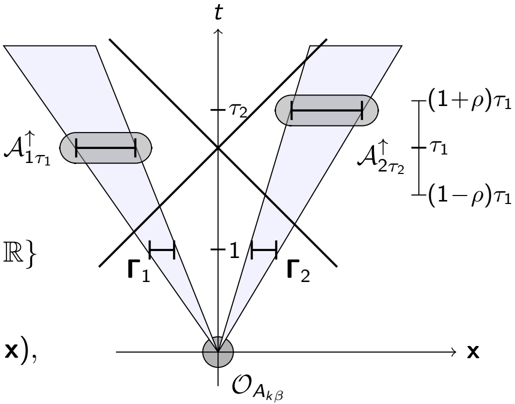
$$f_k(t,\mathbf{x}) = \int \mathrm{d}^3k \, e^{\mathrm{i}\mathbf{k}\cdot\mathbf{x} - \mathrm{i}\omega_m(\mathbf{k})t} \, \tilde{f}_k(\mathbf{k}), \quad \tilde{f}_k \in \mathscr{C}_c^\infty(\mathbb{R}^s), \quad \omega_m(\mathbf{k}) := \sqrt{\mathbf{k}^2 + m^2}$$

- velocity $\mathbf{v}(\mathbf{k}) = \frac{\mathbf{k}}{\omega_m(\mathbf{k})}$
- *velocity support*

  $\mathbf{\Gamma}_f := \mathbf{v}(\operatorname{supp} \tilde{f})$
- *propagation region*

  $\Upsilon_f := \{(t, \mathbf{v}t), \ \mathbf{v} \in \mathbf{\Gamma}_f, \ t \in \mathbb{R}\}$
- *creation operators*

  $\mathcal{A}_{k\tau} = \int \mathrm{d}^3x \, f_k(\tau,\mathbf{x}) \, A_{k\beta}(\tau,\mathbf{x}),$



Lemma: Let f_k be regular s.t. $\mathbf{\Gamma}_1 \cap \mathbf{\Gamma}_2 = \emptyset$ and $A_{k\beta}$ have finite degree.

$$\exists \, \rho > 0 \ \forall \ |\tau_1 - \tau_2| \le \rho \, |\tau_1| : \quad \|[\mathcal{B}_{1\tau_1}, \mathcal{B}_{2\tau_2}]\| \le \frac{C_N \left\|A_{1\beta(\tau_1)}\right\| \left\|A_{2\beta(\tau_2)}\right\|}{1 + |\tau_1|^N + |\tau_2|^N}$$

Assembling the Mathematical Arsenal

The reason why Discrete Cook works may be summarized:

Lemma (local difference estimate) Let $A_{k\beta}$ be RS families of finite degree, and f_k regular positive-energy Klein-Gordon solutions with disjoint velocity supports. Then for sufficiently small scaling $\mu > 0$, $\exists\ \rho > 0\ \forall\ |\tau_1 - \tau_2| \leq \rho\,|\tau_1|$,

$$\|\Psi_{\tau_2} - \Psi_{\tau_1}\|^2 \leq C_1 \sum_{k=1}^{n} \|\mathcal{B}_{k\tau_2}\Omega - \mathcal{B}_{k\tau_1}\Omega\|^2 + C_2\,|\tau_1|^{-\delta}$$

Proof based on **non-equal-time** commutator estimates, **energy-bounds** [Buchholz'90], and **Clustering** arguments from [Dybalski'05], [Buchholz'77], and [Araki, Hepp, Ruelle'62].

Outlook

Summary and Outlook

- Strengthened Reeh-Schlieder useful for Scattering Theory
- Discretized Cook's method improves Convergence,
  but also needs stronger technical tools: In particular,
  Non-Equal-Time Versions of
  - Commutator Estimates
  - Energy-Bounds
  - Clustering Estimates

Open Questions and Next Steps

- Any News for Physical Properties $W^{\pm}$ and S-Matrix?
- Quantitative Results on Reeh-Schlieder?
- Construct Asymptotic Observables (Araki-Haag Detectors)
- Relaxation of Localization Assumption $(A_\beta) \subset \mathfrak{A}(\mathcal{O})$
  - $\mathcal{O} \to \mathcal{O}_{R(\beta)}$ — e.g. with polynomially growing radii
  - $\mathcal{O} \to \mathcal{W}$ — unbounded wedge regions $\mathcal{W}$ appear in context of
    - Polarization-free Generators [Borchers et al'01]
    - non-commutative flat space-times [Grosse, Lechner'07]

Thanks for your attention!

Appendix: Why do we bother?

Wave Operators and S-Matrix

Let $\mathscr{F}$ denote Fock space over finite RS-degree 1-particle vectors and $\mathscr{F}_{\mathrm{disj}} \subset \mathscr{F}$ the set of product states with disjoint $\mathbf{\Gamma}_k$.

Def. (Møller op.) For $\Psi_{\mathrm{prod}} = \Psi_1(f_1)\Omega \otimes \ldots \otimes \Psi_n(f_n)\Omega \in \mathscr{F}_{\mathrm{disj}}$, $\Psi_k = \lim_{\beta\to 0} \tilde{f}_k(\mathbf{P}) A_{k\beta}\Omega$ define

$$W_\pm : \begin{cases} \mathscr{F}_{\mathrm{disj}} \longrightarrow \mathscr{H}, \\ \Psi_{\mathrm{prod}} \longmapsto \lim\limits_{\tau\to\pm\infty} \mathcal{B}_{1\tau}\ldots\mathcal{B}_{n\tau}\Omega. \end{cases}$$

The S-matrix is defined for $\Psi, \Phi \in \mathscr{F}_{\mathrm{disj}}$ by

$$\langle \Psi, S\Phi \rangle := \langle W_+\Psi, W_-\Phi \rangle .$$

Plausibility of **Strengthened Reeh-Schlieder**?

- Rem. There are examples of QFT-models exhibiting $A \in \mathfrak{A}(\mathcal{O})$ which violate the Herbst spectral regularity condition.

- Proposition. In scalar free field theory, there exist Reeh-Schlieder families A_β of arbitrarily small degree $\gamma > 0$.

  Proof. Taking $A_\beta := \phi(f)\mathrm{e}^{-\beta|\phi(f)|^N}$ for compactly supported f has degree $\gamma = 1/N$ for any $N \in 2\mathbb{N}$ does the job.

- Conjecture: $\Psi_1 \in \mathscr{H}_1$ single-particle state with **sufficiently small** Reeh-Schlieder degree $\gamma < 1 \Longrightarrow \Psi_1$ non-interacting.

- Proposition. Assume there is a regular local $A \in \mathfrak{A}(\mathcal{O})$ with Herbst-exponent $\epsilon > 0$. Then one can construct $A_\beta \in \mathfrak{A}(\mathcal{O}+B_\epsilon)$ s.t.

$$\|E(\Delta)(A_\beta\Omega - \Psi_1)\| < C_\Delta \beta, \quad \ln\|A_\beta\| < \beta^{-\gamma}$$

  for any compact $\Delta \subset \mathbb{R}^{s+1}$, with suitable C_Δ, and $\gamma \sim 1/\epsilon$.