Inverse scattering at fixed energy in Kerr-Newman-de Sitter black holes.

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Outline of the talk

- Kerr-Newman-de-Sitter black holes (rotating black holes):
  - Spacetime $(\mathcal{M}, g)$ with g Lorentzian metric.
  - g completely characterized by $M > 0$ (mass), $Q \in \mathbb{R}$ (electric charge), $a \in \mathbb{R}^*$ (angular momentum per unit mass) and $\Lambda > 0$ (cosmological constant).

  "Can we determine g by observing waves at infinities?"

- Massless Dirac fields in KN-dS black holes:
  - Dirac waves: $i\partial_t u = \mathbb{D}u$, $\mathbb{D}$ Dirac operator.
  - Scattering matrix $S_g(\lambda)$, λ a fixed energy.

- Main result:
  - $g \longrightarrow S_g(\lambda)$ is one-to-one for a fixed energy $\lambda \in \mathbb{R}$.
  - Actually, our result is better.

Kerr-Newman-de-Sitter Black Holes

- Spacetime $(\mathcal{M}, g)$:

$$\mathcal{M} = \mathbb{R}_t \times \Sigma, \quad \Sigma =]r_-, r_+[_r \times \mathbb{S}^2_{\theta,\varphi},$$

equipped with a Lorentzian metric

$$\begin{aligned} g \quad = \quad & \frac{\Delta_r - \Delta_\theta a^2 \sin^2\theta}{\rho^2} dt^2 - \frac{2a\sin^2\theta}{E\rho^2}(\Delta_r - \Delta_\theta a^2 \sin^2\theta) dt d\varphi \\ & - \frac{\rho^2}{\Delta_r} dr^2 - \frac{\rho^2}{\Delta_\theta} d\theta^2 - \frac{\sin^2\theta}{E^2\rho^2}\left(\Delta_\theta (r^2+a^2)^2 - \Delta_r a^2 \sin^2\theta\right) d\varphi^2. \end{aligned}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad E = 1 + \frac{a^2\Lambda}{3} > 0, \quad \Delta_\theta = 1 + \frac{a^2\Lambda\cos^2\theta}{3} > 0,$$

$$\Delta_r = (r^2+a^2)(1 - \frac{\Lambda r^2}{3}) - 2Mr + Q^2,$$

The event and cosmological horizons

- Singularities: Δ_r possesses 4 roots: $r_n < 0 < r_c < r_- < r_+$ if

$$
\begin{array}{ll}
(1) & \frac{a^2\Lambda}{3} \leq 7 - 4\sqrt{3}, \\
(2) & M^-_{crit} < M < M^+_{crit},
\end{array}
$$

with

$$
M^{\pm}_{crit} = \frac{1}{\sqrt{18\Lambda}} \left(\left(1 - \frac{a^2\Lambda}{3}\right) \pm \sqrt{\left(1 - \frac{a^2\Lambda}{3}\right)^2 - 4\Lambda(a^2 + Q^2)} \right)^2 \left(2\left(1 - \frac{a^2\Lambda}{3}\right)^2 \mp \sqrt{\left(1 - \frac{a^2\Lambda}{3}\right)^2 - 4\Lambda(a^2 + Q^2)} \right).
$$

- Cosmological and event horizons: $\{r = r_{\pm}\}$

$$
\boxed{\Delta_r > 0 \text{ in } \{r_- < r < r_+\}}
$$

Properties of KN-dS black holes

- Symmetries:
  - Cylindrical symmetry (∂_φ Killing vector field).
  - Time-translation (∂_t Killing vector field).

- Nullcones: $\partial_r, \partial_\theta, \partial_\varphi$ everywhere spacelike in $\mathcal{M}$.
  ∂_t everywhere timelike except on the two ergospheres

  $$\mathcal{E}^\pm = \{(r,\theta) \in \mathcal{M},\ \Delta_r - \Delta_\theta a^2 \sin^2\theta < 0\},$$

  where it becomes spacelike.

- Lack of stationarity: there exists no globally defined timelike Killing vector field on $\mathcal{M}$.

Properties of Kerr-Newman-de-Sitter black holes

- stationary observers: observers living on the worldlines

$$r = const,\ \theta = const,\ \varphi = \omega t + const,$$

with $\omega = const$. For such observers, if $r_- << r << r_+$, then the variable t corresponds to their proper time.

- Principal null geodesics: the spacetime $\mathcal{M}$ is foliated by the two families of incoming and outgoing principal null geodesics generated by

$$V^\pm = \frac{r^2 + a^2}{\Delta_r}\left(\partial_t + \frac{aE}{r^2+a^2}\partial_\varphi\right) \pm \partial_r.$$

The principal null geodesics do not reach $\{r = r_\pm\}$ in a finite time t! The horizons are perceived as asymptotic regions by stationary observers.

A new radial variable

- Tortoise radial variable: $\{r = r_\pm\} \leftrightarrow \{x = \pm\infty\}$.

$\frac{dx}{dr} = \frac{r^2+a^2}{\Delta_r}$

$$x = \frac{1}{2\kappa_-}\ln(r-r_-)+\frac{1}{2\kappa_+}\ln(r_+-r)+\frac{1}{2\kappa_c}\ln(r-r_c)+\frac{1}{2\kappa_n}\ln(r-r_n)+c,$$

  - c is any constant of integration.
  - $\kappa_j = \frac{\Delta_r'(r_j)}{2(r_j^2+a^2)}$, $j = -, +, c, n$.

- New framework: $\mathcal{B} = \mathbb{R}_t \times \mathbb{R}_x \times \mathbb{S}^2_{\theta,\varphi}$,

$$\begin{aligned} g \quad = \quad & \frac{\Delta_r}{\rho^2}\left[dt - \frac{a\sin^2\theta}{E}d\varphi\right]^2 - \frac{\rho^2\Delta_r}{(r^2+a^2)^2}dx^2 - \frac{\rho^2}{\Delta_\theta}d\theta^2 \\ & - \frac{\Delta_\theta\sin^2\theta}{\rho^2}\left[a\,dt - \frac{r^2+a^2}{E}d\varphi\right]^2. \end{aligned}$$

Massless Dirac fields in KN-dS black holes

Dirac fields: $\psi \in \mathcal{H} = L^2(\mathbb{R} \times \mathbb{S}^2, dxd\theta d\varphi; \mathbb{C}^2)$ and satisfies

$$i\partial_t \psi = H\psi,$$

where

$$H = J^{-1} H_0, \qquad H_0 = \Gamma^1 D_x + a(x) H_{\mathbb{S}^2} + c(x, D_\varphi)$$

- $D_x = -i\partial_x$, $D_\theta = -i\partial_\theta$, $D_\varphi = -i\partial_\varphi$.
- Angular Dirac operator:

$$H_{\mathbb{S}^2} = \sqrt{\Delta_\theta} \left[\Gamma^2 D_\theta + \Gamma^2 \frac{i\Lambda a^2 \sin(2\theta)}{12\Delta_\theta} + \frac{\Gamma^3}{\sin\theta} D_\varphi + \Gamma^3 \frac{\Lambda a^2 \sin(\theta)}{3\Delta_\theta} D_\varphi \right].$$

- Dirac matrices: $\Gamma^i \Gamma^j + \Gamma^j \Gamma^i = 2\delta_{ij} I_2, \quad \forall i, j = 1, 2, 3.$

$$\Gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Gamma^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma^3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Massless Dirac fields in KN-dS black holes

- Dirac equation:

$$i\partial_t \psi = H\psi, \quad H = J^{-1} H_0, \qquad H_0 = \Gamma^1 D_x + a(x) H_{\mathbb{S}^2} + c(x, D_\varphi).$$

- Potentials:

$$a(x) = \frac{\sqrt{\Delta_r}}{r^2 + a^2}, \quad c(x, D_\varphi) = \frac{aE}{r^2 + a^2} D_\varphi + \frac{qQr}{r^2 + a^2}.$$

$$J = I_2 + a(x) b(\theta) \Gamma^3, \quad b(\theta) = \frac{a \sin\theta}{\sqrt{\Delta_\theta}},$$

$$J^{-1} = (1 - a(x)^2 b(\theta)^2)(I_2 - a(x) b(\theta) \Gamma^3).$$

- Cylindrical symmetry: The equation is simplified by decomposing it onto the angular modes $\{e^{ik\varphi}\}$, $k \in \frac{1}{2} + \mathbb{Z}$.

$$D_\varphi \longleftrightarrow k.$$

Asymptotics of the potentials

We recall that the constants $\kappa_\pm \in \mathbb{R}^\mp$.

- Asymptotics of $a(x)$:

$$a(x) \sim a_\pm e^{\kappa_\pm x}, \ x \to \pm\infty.$$

- Asymptotics of $c(x,k)$, ($c(x,k)$ is a long range potential)

$$c(x,k) \sim \frac{aEk + qQr_\pm}{r_\pm^2 + a^2} + c_\pm e^{2\kappa_\pm x}, \ x \to \pm\infty,$$

- Asymptotics of $J(x,.)$:

$$\sup_{\theta \in [0,\pi]} \|J(x,.) - I_2\|_\infty = O\left(e^{\kappa_\pm x}\right), \ x \to \pm\infty.$$

Scattering theory

There are two usual and different ways to define the scattering matrix $S(\lambda)$ where $\lambda \in \mathbb{R}$ is the energy. Roughly speaking :

- The time-dependent approach:
  - We define the time-dependent wave operators (in a two Hilbert settings here) and the scattering operator S.
  - We diagonalize the "free Hamiltonian" with a unitary operator $\mathcal{F}$ and define the scattering matrix $S(\lambda) = \mathcal{F} S \mathcal{F}^*$, $\lambda \in \mathbb{R}$.
- The stationary approach (discussed in this talk):
  - We solve the stationary equation $H\psi = \lambda\psi$ using the separability of the Dirac equation.
  - The system of angular ODEs permits to define generalized spherical harmonics and to decompose the full scattering matrix $S(\lambda)$ onto reduced scattering matrices $S_{kl}(\lambda)$.
  - Some special solutions - called the Jost solutions - of the system of radial ODES encode the reduced scattering matrices $S_{kl}(\lambda)$.

Theorem

For KN-dS black holes, the above definitions are equivalent.

The stationary equation

$$
\begin{aligned}
H\psi = \lambda\psi \quad &\Longleftrightarrow \quad J^{-1}(H_0 - \lambda J)\psi = 0, \\
&\Longleftrightarrow \quad \left[\Gamma^1 D_x + c(x, D_\varphi) - \lambda + \ a(x)(H_{\mathbb{S}^2} - \lambda b(\theta)\Gamma^3)\right]\psi = 0,
\end{aligned}
$$

with $\psi \in \mathcal{H} = L^2(\mathbb{R} \times \mathbb{S}^2; \mathbb{C}^2)$.

Theorem

The operators D_φ and $A_{\mathbb{S}^2}(\lambda) = H_{\mathbb{S}^2} - \lambda b(\theta)\Gamma^3$ possess a common basis of eigenfunctions $Y_{kl}(\lambda) \in L^2(\mathbb{S}^2, \mathbb{C}^2)$:

$$
\begin{aligned}
A_{\mathbb{S}^2}(\lambda) Y_{kl}(\lambda) &= \mu_{kl}(\lambda) Y_{kl}(\lambda), \\
D_\varphi Y_{kl}(\lambda) &= k Y_{kl}(\lambda).
\end{aligned}
$$

The stationary equation

Also, we can write (separation of variables)

$$\mathcal{H} = \bigoplus_{(k,l)\in(1/2+\mathbb{Z})\times\mathbb{N}^*} \mathcal{H}_{kl}(\lambda),$$

$$\mathcal{H}_{kl}(\lambda) = L^2(\mathbb{R};\mathbb{C}^2) \otimes Y_{kl}(\lambda) \ \simeq \ L^2(\mathbb{R},\mathbb{C}^2),$$

such that, if $\psi = \sum_{k,l} \psi_{kl}(x) \otimes Y_{kl}(\lambda)$, the stationary equation $H\psi = \lambda\psi$, is equivalent to the countable family of one-dimensional (radial) stationary equations

$$H_{kl}(\lambda)\psi_{kl}(x) = \lambda\,\psi_{kl}(\lambda),$$

where

$$H_{kl}(\lambda) = \Gamma^1 D_x + \mu_{kl}(\lambda)a(x)\Gamma^2 + c(x,k).$$

We emphasize that $Y_{kl}(\lambda)$ and $\mu_{kl}(\lambda)$ depend on the parameters of the black hole we are looking for !

Reduction to the short range case.

$$\left[\Gamma^1 D_x + \mu_{kl}(\lambda) a(x)\Gamma^2 + \underbrace{c(x,k)}_{\text{long-range}}\right]\psi = \lambda\psi,$$

For $k \in 1/2 + \mathbb{Z}$, we define the unitary operator

$$U_k = e^{-iC(x,k)\Gamma^1}, \quad C(x,k) = \int_{-\infty}^{x} [c(s,k) - \Omega_-(k)]ds + \Omega_-(k)x,$$

where $\Omega_\pm(k) = \frac{aEk+qQr_\pm}{r_\pm^2+a^2}$. The spinor $\phi = U_k^{-1}\psi$ satisfies

$$\left[\Gamma^1 D_x + \mu_{kl}(\lambda) V_k(x)\right]\phi = \lambda\phi,$$

where

$$V_k(x) = \begin{pmatrix} 0 & a(x)e^{2iC(x,k)} \\ a(x)e^{-2iC(x,k)} & 0 \end{pmatrix}.$$

Note that $V_k(x)$ now decays exponentially at both horizons $\{x = \pm\infty\}$.

The Jost functions

Set $z = -\mu_{kl}(\lambda) \in \mathbb{R}$. The stationary equation

$$\left[\Gamma^1 D_x - zV_k(x)\right] \phi = \lambda\phi,$$

possesses 2×2 matrix solutions $F_L(x, \lambda, z)$ and $F_R(x, \lambda, z)$ called Jost functions that have the asymptotics:

$$\begin{aligned} F_L(x, \lambda, k, z) &= e^{i\Gamma^1 \lambda x}(I_2 + o(1)), \ x \to +\infty, \\ F_R(x, \lambda, k, z) &= e^{i\Gamma^1 \lambda x}(I_2 + o(1)), \ x \to -\infty. \end{aligned}$$

- F_L and F_R are fundamental matrices of the stationary equation, i.e. $\det F_{L/R} = 1$.
- $\exists\ A_L(\lambda, k, z) = \begin{bmatrix} a_{L1}(\lambda, k, z) & a_{L2}(\lambda, k, z) \\ a_{L3}(\lambda, k, z) & a_{L4}(\lambda, k, z) \end{bmatrix}$ such that

$$F_R(x, \lambda, k, z) = F_L(x, \lambda, k, z) A_L(\lambda, k, z).$$

Stationary representation of $S(\lambda)$

The matrix $A_L(\lambda, k, z)$ encodes the scattering properties associated to the stationary equation

$$\left[\Gamma^1 D_x - zV_k(x)\right]\phi = \lambda\phi.$$

We define the simplified and reduced scattering matrix by

$$\hat{S}(\lambda, k, z) = \left[\begin{array}{cc} \hat{T}(\lambda, k, z) & \hat{R}(\lambda, k, z) \\ \hat{L}(\lambda, k, z) & \hat{T}(\lambda, k, z) \end{array}\right],$$

where

$$\hat{T}(\lambda, k, z) = \frac{1}{a_{L1}(\lambda, k, z)}, \quad \hat{R}(\lambda, k, z) = -\frac{a_{L2}(\lambda, k, z)}{a_{L1}(\lambda, k, z)},$$

$$\hat{L}(\lambda, k, z) = \frac{a_{L3}(\lambda, k, z)}{a_{L1}(\lambda, k, z)}.$$

Stationary representation of $S(\lambda)$

The physical and global scattering matrix $S(\lambda)$ is given in terms of simplified and reduced scattering matrices by

$$S(\lambda) = \oplus_{k,l} S_{kl}(\lambda) = \oplus_{k,l} \begin{bmatrix} T_{kl}(\lambda) & R_{kl}(\lambda) \\ L_{kl}(\lambda) & T_{kl}(\lambda) \end{bmatrix},$$

where

$$\begin{aligned} T_{kl}(\lambda) &= e^{-i\beta(k)} \hat{T}(\lambda, k, \mu_{kl}(\lambda)), \\ R_{kl}(\lambda) &= e^{-2i\beta(k)} \hat{R}(\lambda, k, \mu_{kl}(\lambda)), \\ L_{kl}(\lambda) &= \hat{L}(\lambda, k, \mu_{kl}(\lambda)), \end{aligned}$$

with

$$\beta(k) = \int_{-\infty}^{0} [c(s,k) - \Omega_-(k)]\, ds + \int_{0}^{+\infty} [c(s,k) - \Omega_+(k)]\, ds.$$

Scattering matrix : rearrangement

At this stage, we have constructed for all $\lambda \in \mathbb{R}$, the scattering matrix $S(\lambda)$ as a unitary operator on $L^2(\mathbb{S}^2; \mathbb{C}^2)$.

Using the cylindrical symmetry and the matrix structure of the scattering matrix, $S(\lambda)$ can be expressed as

$$S(\lambda) = \oplus_{k \in \frac{1}{2}+\mathbb{Z}} S_k(\lambda), \quad S_k(\lambda) = \left[\begin{array}{cc} T_k^L(\lambda) & R_k(\lambda) \\ L_k(\lambda) & T_k^R(\lambda) \end{array} \right],$$

where $T_k^{L/R}(\lambda), R_k(\lambda)$ and $L_k(\lambda)$ act on $L^2((0,\pi), d\theta; \mathbb{C})$ and correspond to the transmission and reflection operators of our scattering experiment.

We can state now the main uniqueness result of this work :

An inverse result at fixed energy

Theorem

Let (M, Q^2, a, Λ) *and* $(\tilde{M}, \tilde{Q}^2, \tilde{a}, \tilde{\Lambda})$ *be the parameters of two a priori different KN-dS black holes. Denote by* $S(\lambda)$ *and* $\tilde{S}(\lambda)$ *the corresponding scattering matrices at a fixed energy* $\lambda \in \mathbb{R}$*. Assume that one of the following equalities is fulfilled*

$$\begin{aligned} R_k(\lambda) &= \tilde{R}_k(\lambda), \\ L_k(\lambda) &= \tilde{L}_k(\lambda), \end{aligned}$$

as operators on $\mathcal{L} = L^2((0,\pi);\mathbb{C})$ *and for two different values of* $k \in \frac{1}{2} + \mathbb{Z}$*. Then the parameters of the two black holes coincide, i.e.*

$$M = \tilde{M}, \ a = \tilde{a}, \ Q^2 = \tilde{Q}^2, \ \Lambda = \tilde{\Lambda}.$$

Comment: a more general result

We obtain more than only 4 parameters. Precisely,

Theorem

Under the same assumptions as in the previous Theorem, we recover in fact the function

$$\frac{\lambda - c(x,k)}{a(x)},$$

up to a diffeomorphism. Then, from the explicit forms of the potentials, we obtain $M = \tilde{M},\ a = \tilde{a},\ Q^2 = \tilde{Q}^2,\ \Lambda = \tilde{\Lambda}$.

In the particular case $Q = 0$ *or if the scattering operators are known for two different energies* $\lambda \in \mathbb{R}$, *we get more precise results. Precisely, there exists a constant* $\sigma \in \mathbb{R}$ *such that*

$$\begin{aligned} \tilde{a}(x) &= a(x-\sigma), \\ \tilde{c}(x,k) &= c(x-\sigma,k). \end{aligned}$$

Comment: possible extension

Consider the class of Lorentzian metrics

$$g = T^2 \left[\frac{W^2}{Z}(dt + md\varphi)^2 - \frac{Z}{W^2}dr^2 - \frac{Z}{X^2}d\mu^2 - \frac{X^2}{Z}(adt + pd\varphi)^2 \right],$$

where

- $T^2 = T^2(r, \mu) > 0.$
- $W^2 = W^2(r) > 0, \quad X^2 = X^2(\mu) > 0.$
- $m = m(\mu), \quad p = p(r).$
- $Z(r, \mu) = p(r) - am(\mu), \quad a = \text{constant}, \quad \mu = \cos(\theta).$

These Lorentzian metrics are stationary axisymmetric and possess a pair of shearfree geodesic null congruences. Moreover, their geodesic flow is completely integrable.

(Carter 1968, Debever, Kamran, McLenaghan 1983) The wave equation is separable on (M, g). The Klein-Gordon equation is separable on (M, g) iff $T^2 = 1$.

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Comment: possible extension

(Debever, Kamran, McLenaghan, 1984) If we assume additionally that (M, g) has type D in the Petrov classification, *i.e.* the one-form

$$\omega = \frac{1}{4Z}(m'(\mu)dr + ap'(r)d\mu),$$

is closed, then the massless Dirac equation is separable on (M, g).

- This condition implies that $m(\mu)$ and $p(r)$ must be polynomial of degree 2. For instance, $p(r) = r^2 + a^2$ and $m(\mu) = a(1 - \mu^2)$.
- We recover the family of Kerr-dS-TAUB-NUT spacetime by setting

$$m(\mu) = a(1-\mu^2)+2l(1-\mu), \quad p(r) = r^2+(a+l)^2, \quad Z^2 = r^2+(a\mu+l)^2,$$

$$W^2 = (a^2-l^2+e^2+g^2)-2Mr+r^2-\Lambda\left((a^2 - l^2)l^2 + (\frac{a^2}{3} + 2l^2)r^2 + \frac{r^4}{3}\right)$$

$$X^2 = (1 - \mu^2)\left(1 + \frac{4}{3}\Lambda a l \mu + \frac{\Lambda}{3}a^2\mu^2\right).$$

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$$W^2 = (a^2-l^2+e^2+g^2)-2Mr+r^2-\Lambda\left((a^2 - l^2)l^2 + (\frac{a^2}{3} + 2l^2)r^2 + \frac{r^4}{3}\right)$$

$$X^2 = (1 - \mu^2)\left(1 + \frac{4}{3}\Lambda al\mu + \frac{\Lambda}{3}a^2\mu^2\right).$$

Comment: possible extension

Under the assumptions:

(1) (Radial part) There exist $0 < r_- < r_+ < \infty$ such that

- for all $r_- < r < r_+$, $W^2(r) > 0$,
- $W^2(r_\pm) = 0$ and $(W^2)'(r_\pm) \neq 0$,
- $W^2 \in C^2$.

(2) (Angular part) The angular function $X(\mu)$ should be a small perturbation of

- $X \in C^\infty(0, B)$, $X > 0$, $X(0) = X(B) = 0$,
- X has a unique non-degenerate maximum at $\mu_0 \in (0, B)$.

(3) For all $r \in (r_-, r_+)$, $\mu \in (0, B)$, we impose

$$\frac{X(\mu)}{m(\mu)} > \frac{W(r)}{p(r)}.$$

The previous uniqueness results could be generalized to (M, g) (work in progress with Alexei Iantchenko).

References

The Dirac equation $i\partial_t\psi = H\psi$ can be understood as an evolution equation on the manifold $\Sigma = \mathbb{R} \times \mathbb{S}^2$ having two different ends - the event $\{x = -\infty\}$ and cosmological $\{x = +\infty\}$ horizons - which are asymptotically hyperbolic.

- Joshi, Sa Barreto (Acta Math. [2000]): "asymptotics of the metric uniquely determined from the knowledge of $S(\lambda)$ (associated to the laplacian) at a fixed energy $\lambda \in \mathbb{R}^+$ outside a discret set".
- Sa Barreto (Duke Math. J. [2005]): "metric uniquely determined from the knowledge of $S(\lambda)$ (associated to the laplacian) for all $\lambda \in \mathbb{R}^+$ except on a discrete set of energies".
- Isozaki H., Kurylev J. (Mathematical Society of Japan Memoirs, 2014): "Introduction to spectral theory and inverse problem on asymptotically hyperbolic manifolds".

Main ideas of the proof.

Recall that $Y_{kl} = (Y_{kl}^1, Y_{kl}^2)$ are the eigenfunctions of the angular operator $A_{\mathbb{S}^2}(\lambda)$.

Proposition

The operators $R_k(\lambda) R_k(\lambda)^*$, (resp. $R_k(\lambda)^* R_k(\lambda)$), on $L^2((0,\pi))$ are diagonalizable on the Hilbert basis of eigenfunctions $(Y_{kl}^1)_{l \in \mathbb{N}^*}$, (resp. $(Y_{kl}^2)_{l \in \mathbb{N}^*}$), associated to the eigenvalues $|R_{kl}(\lambda)|^2$.
Moreover, the map $l \mapsto |R_{kl}(\lambda)|$ is strictly increasing for l large enough (technical point).

From the equality $R_k(\lambda) = \tilde{R}_k(\lambda)$, we can deduce that

$$\exists L > 0,\ \forall l \geq L, \quad R_{kl}(\lambda) = \tilde{R}_{kl}(\lambda), \quad Y_{kl}^j = \tilde{Y}_{kl}^j,\ j = 1, 2,$$

up to multiplicative constants of modulus 1.

Main ideas of the proof : the Frobenius method.

Proposition

Set $\zeta = \frac{a^2\Lambda}{3}$. *For all* $\lambda \in \mathbb{R}$ *and* $(k,l) \in (\frac{1}{2} + \mathbb{N}) \times \mathbb{N}^*$, *there exist constants* $c_{kl}^{\lambda} \in \mathbb{C}$ *such that when* $\theta \to 0$

$$
\begin{aligned}
Y_{kl}(\theta,\varphi) \quad &= \quad c_{kl}^{\lambda} e^{ik\varphi} \Bigg\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \theta^k + \frac{i\mu_{kl}(\lambda)}{(2k+1)\sqrt{1+\zeta}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \theta^{k+1} \\
&+ \quad \left[\frac{k}{6} + \frac{\zeta}{2(1+\zeta)} + \frac{\zeta k - a\lambda}{1+\zeta} - \frac{\mu_{kl}(\lambda)^2}{(2k+1)(1+\zeta)} \right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{\theta^{k+2}}{2} \\
&+ \quad O(\theta^{k+3}) \Bigg\}.
\end{aligned}
$$

From $Y_{kl}^j = \tilde{Y}_{kl}^j$, $j = 1,2$, and for two different k, we deduce that $a = \tilde{a}$, $\Lambda = \tilde{\Lambda}$. In particular,

$$\forall k \in \frac{1}{2} + \mathbb{Z},\ \forall l \in \mathbb{N}^*, \quad \mu_{kl}(\lambda) = \tilde{\mu}_{kl}(\lambda).$$

Complexification of the angular momentum.

So, it remains to recover the mass M and the charge Q.

- We allow the physical angular momenta $\mu_{kl}(\lambda)$ to be complex. We set

$$z = -\mu_{kl}(\lambda).$$

- The Jost functions $F_L(x, \lambda, k, z)$ and $F_R(x, \lambda, k, z)$ extend analytically to $\mathbb{C}$ with respect to z.
- Similarly the scattering data $A_L(\lambda, k, z)$ extend analytically to $\mathbb{C}$ with respect to z. Moreover, the entries of the matrix $A_L(\lambda, k, z)$ satisfy:

Lemma

$$z \to a_{Lj}(\lambda, k, z) \in H(\mathbb{C}), \quad |a_{Lj}(\lambda, k, z)| \leq e^{A|Rez|}, \quad A = \int_{\mathbb{R}} a(x)dx.$$

Nevanlinna class

Theorem (Nevanlinna class, Uniqueness)

A function f belongs to $N(\Pi^+)$, where $\Pi^+ = \{z \in \mathbb{C} : \ Re(z) > 0\}$, if it is analytic on Π^+ and if

$$\sup_{0<r<1} \int_{-\pi}^{\pi} \ln^+ \left| f\left(\frac{1-re^{i\varphi}}{1+re^{i\varphi}}\right)\right| d\varphi < \infty,$$

where $\ln^+(x) = \begin{cases} \ln x, & \ln x \geq 0, \\ 0, & \ln x < 0. \end{cases}$

If $f \in N(\Pi^+)$ *satisfies* $f(\alpha_l) = 0$ *with* $\displaystyle\sum_{l=0}^{+\infty} \frac{1}{\alpha_l} = \infty$*, then* $f \equiv 0$ *in* $\mathbb{C}$*.*

Application

Proposition

The scattering data $a_{Lj}(\lambda, k, z) \in N(\Pi^+)$.

Corollary

Under our main assumption, we have (up to multiplicative constants of modulus 1*)*

$$a_{Lj}(\lambda, k, \mu_{kl}(\lambda)) = \tilde{a}_{Lj}(\lambda, k, \tilde{\mu}_{kl}(\lambda)), \forall l \in \mathbb{N}^*.$$

Since $\mu_{kl}(\lambda) = \tilde{\mu}_{kl}(\lambda)$ *and* $\displaystyle\sum_{l=0}^{\infty} \frac{1}{\mu_{kl}(\lambda)} = \infty$*, we get*

$$a_{Lj}(\lambda, k, z) = \tilde{a}_{Lj}(\lambda, k, z), \forall z \in \mathbb{C},$$

(up to multiplicative constants of modulus 1*)*

An inverse result at localized energy

Corollary

Assume that $R_k(\lambda) = \tilde{R}_k(\lambda)$ for all λ in an open interval I. Then the potentials $a(x)$ and $c(x,k)$ are uniquely determined.

Proof: From the assumption, we can show for instance that

$$a_{L2}(\lambda, k, z) = \alpha \tilde{a}_{L2}(\lambda, k, z), \quad \forall z \in \mathbb{C}, \ \forall \lambda \in I,$$

where $|\alpha| = 1$. This implies that

$$\hat{q}(., k)(2\lambda) = \alpha \hat{\tilde{q}}(., k)(2\lambda), \quad \forall \lambda \in I,$$

where $q(x,k) = e^{2iC(x,k)} a(x)$ is exponentially decreasing on $\mathbb{R}$. Hence

$$\hat{q}(., k)(2\lambda) = \alpha \hat{\tilde{q}}(., k)(2\lambda), \quad \forall \lambda \in \mathbb{R},$$

and therefore

$$q(x,k) = \alpha \tilde{q}(x), \quad \forall x \in \mathbb{R}.$$

An inverse result at localized energy

Taking the logarithmic derivative with respect to x, we obtain,

$$\frac{a'(x)}{a(x)} + 2ic(x,k) = \frac{\tilde{a}'(x)}{\tilde{a}(x)} + 2i\tilde{c}(x,k).$$

Thus, taking the real and imaginary parts of this equality, we have

$$a(x) = \tilde{a}(x), \quad c(x,k) = \tilde{c}(x,k).$$

End of the proof of the inverse problem at fixed energy

"Up to a Liouville transformation in the variable x", we define the 2×2 matrix-valued function $P(x, \lambda, k, z)$ by

$$P(x, \lambda, k, z)\tilde{F_R}(x, \lambda, k, z) = F_R(x, \lambda, k, z).$$

Question : What can we say about $P(x, \lambda, k, z)$?

- By inverting $\tilde{F_R}$, $(\det(\tilde{F_R}) = 1)$, $z \longrightarrow P_j(x, \lambda, k, z)$ belong to $H(\mathbb{C})$, are of exponential type and are bounded on $i\mathbb{R}$.
- We calculate the asymptotics of $a_{Lj}(\lambda, k, z)$, $z \to +\infty$.
- Algebraic manipulations + uniqueness of the $a_{Lj}(\lambda, k, z)$, we can show that $z \longrightarrow P_j(x, \lambda, k, z)$ are also bounded on $\mathbb{R}$.
- Phragmen-Lindelöf Thm: $z \longrightarrow P_j(x, \lambda, k, z)$ are bounded on $\mathbb{C}$.

End of the proof

- Liouville Thm: $P_j(x, \lambda, k, z) = P_j(x, \lambda, k, 0)$ for all $z \in \mathbb{C}$.
- We calculate explicitly $P_j(x, \lambda, k, 0)$.
- Putting this last result in

$$P(x, \lambda, k, 0)\tilde{F_R}(x, \lambda, k, z) = F_R(x, \lambda, k, z).$$

  we find a simple link between $\tilde{F_R}(x, \lambda, k, z)$ and $F_R(x, \lambda, k, z)$.
- Thus, we can recover (up to a diffeomorphism due to the Liouville transformation) the function

$$\frac{\lambda - c(x, k)}{a(x)},$$

- From the explicit forms of the potentials : uniqueness of the parameters of the black hole.