Boundary conditions and ground states for a scalar field theory on BTZ spacetime

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Quantum fields, scattering and spacetime horizons: mathematical challenges

Les Houches – 25/05/2018

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Outline of the Talk

1. AdS and BTZ: Geometric Data
2. Scalar Fields and Boundary Conditions in AdS
3. Scalar Fields and Boundary Conditions in BTZ
4. A glimpse on an existence theorem for fundamental solutions

References

Brute Force Approach:

- C.D. and H. Ferreira – Phys. Rev. D **94** (2016) no.12, 125016,
- F. Bussola, C. D., H. R. C. Ferreira and I. Khavkine, Phys. Rev. D **96** (2017) no.10, 105016,
- C.D., H. Ferreira, Benito A. Juárez-Aubry, Phys. Rev. D **97** (2018) no.8, 085022
- C. D., H. Ferreira and A. Marta, arXiv:1805.03135 [hep-th].

Algebraic version:

- Marco Benini, C. D., A. Schenkel, arXiv:1712.06686 [math-ph], to appear on Ann. Henri Poinc.
- C.D., H. Ferraira, Rev. Math. Phys. **30** (2018) 0004

Existence and uniqueness of fundamental solutions (funct. anal. aspects)

- C.D., Nicolò Drago, arXiv:1804.03434 [math-ph]

AdS & BTZ - Reasons

We consider the problem of the quantization of a massive scalar field on AdS_{d+1}/BTZ. Why?

1. AdS_{d+1} is a $(d+1)$-dimensional, maximally symmetric solution of the Einstein's equations with *negative* cosmological constant.
2. AdS_{d+1} and BTZ are **not** globally hyperbolic
3. AdS_{d+1} and BTZ possess a conformal boundary,
4. Excellent prototypes to understand (A)QFT in presence of boundaries

Goal: Understand the interplay between boundary conditions and quantization

First step: Construct the fundamental solutions for $\Box - m_0^2 - \xi R$

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AdS - Geometry and the Poincaré chart

AdS_{d+1} can be realized as the locus in $\mathbb{R}^{d+2}$ with metric

$$ds^2 = -dX_0^2 - dX_1^2 + \sum_{i=2}^{d+1} dX_i^2 \,,$$

with

$$-X_0^2 - X_1^2 + \sum_{i=2}^{d+1} X_i^2 = -\ell^2 \,, \qquad \ell^2 \doteq -\frac{d(d-1)}{\Lambda}$$

The Poincaré *chart* is

$$\begin{cases} X_0 = \ell \left(\dfrac{1+z^2}{2z} + \dfrac{-t^2 + \delta^{ij} x_i \, x_j}{2z} \right) \,, \\ X_1 = \dfrac{\ell}{z} t \,, \\ X_i = \dfrac{\ell}{z} x_i \,, \quad i = 2, ..., d, \\ X_{d+1} = \ell \left(\dfrac{1-z^2}{2z} - \dfrac{-t^2 + \delta^{ij} x_i \, x_j}{2z} \right) \,. \end{cases}$$

The Poincaré patch

In the Poincaré chart, PAdS_{d+1}, the metric reads

$$ds^2 = \frac{\ell^2}{z^2}[-dt^2 + dz^2 + \delta^{ij} dx_i dx_j], \quad i, j = 1, ..., d-1$$

where $z > 0$. Observe that

1. PAdS_{d+1} is conformally related to the upper half plane $(\mathring{\mathbb{H}}^{d+1}, \eta)$ with conformal factor $\Omega = \frac{z}{\ell}$.
2. The chordal distance σ in PAdS_{d+1} is $(\ell = 1)$

$$\cosh(\sqrt{2\sigma}) = 1 + \sigma_{\mathbb{R}^{d+2}}, \quad \sigma_{\mathbb{R}^{d+2}}(x, x') \doteq \frac{1}{2} g^{AB}(X_A - X'_A)(X_B - X'_B),$$

where $g^{AB} = diag(-1, -1, 1, ..., 1)$, $A, B = 0, ..., d+1$.

The hyperplane $z = 0$ in $\mathbb{R}^{d+1}$ is the conformal boundary of PAdS_{d+1}.

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The BTZ black hole

It is a **stationary, axi-symmetric** (2+1)-dimensional solution of Einstein's equations with $\Lambda = -\ell^{-2} < 0$ with line element

$$ds^2 = -N^2 dt^2 + \frac{dr^2}{N^2} + r^2(d\varphi + N^\varphi dt)^2,$$

with

$$N^2 = -M + \frac{r^2}{\ell^2} + \frac{J^2}{4r^2}, \quad N^\varphi = -\frac{J}{2r^2}.$$

1. BTZ is topologically $\mathbb{R} \times I \times \mathbb{S}^1$, with $I \subset (0, \infty)$
2. It possesses an inner and an outer horizons at

$$r_\pm^2 = \frac{\ell^2}{2}\left(M \pm \sqrt{M^2 - \frac{J^2}{\ell^2}}\right), \quad |J| \leq M\ell$$

3. It possesses a timelike Killing field $(r > r_+)$

$$\chi = \partial_t + \Omega_{\mathcal{H}} \partial_\varphi, \quad \Omega_{\mathcal{H}} = \frac{r_-}{\ell r_+}.$$

From AdS_3 to BTZ

The BTZ black hole is locally isometric to AdS_3.

The *universal cover* of AdS_3 can be covered with three patches:

$$\text{Region i)} \begin{cases} X_0 = \sqrt{\alpha(r)}\cosh(r_+\varphi - r_- t), & X_1 = \sqrt{\alpha(r)-1}\sinh(r_+ t - r_-\varphi) \\ X_2 = \sqrt{\alpha(r)}\sinh(r_+\varphi - r_- t), & X_3 = \sqrt{\alpha(r)-1}\cosh(r_+ t - r_-\varphi) \end{cases},$$

for $r \geq r_+$ while, for $r_- \leq r \leq r_+$

$$\text{Region ii)} \begin{cases} X_0 = \sqrt{\alpha(r)}\cosh(r_+\varphi - r_- t), & X_1 = -\sqrt{\alpha(r)-1}\sinh(r_+ t - r_-\varphi) \\ X_2 = \sqrt{\alpha(r)}\sinh(r_+\varphi - r_- t), & X_3 = -\sqrt{\alpha(r)-1}\cosh(r_+ t - r_-\varphi) \end{cases}.$$

where $\alpha(r) \doteq \frac{r^2 - r_-^2}{r_+^2 - r_-^2}$, $\varphi \in \mathbb{R}$ and $t \in \mathbb{R}$.

$$\text{BTZ} \iff \varphi \simeq \varphi + 2\pi$$

Klein-Gordon field in PAdS_{d+1}

Consider $\phi : \mathrm{PAdS}_{d+1} \to \mathbb{R}$

$$P\phi = (\Box_{\mathrm{PAdS}} - m_0^2 - \xi R)\phi = 0 \quad \xi \in \mathbb{R} \text{ and } R = -d(d+1)$$

- *Conformal rescaling* $\longrightarrow$ $\Phi \doteq \Omega^{\frac{1-d}{2}}\phi : \mathring{\mathbb{H}}^{d+1} \to \mathbb{R}$ obeys

$$P_\eta \Phi \doteq \left(\Box_\eta - \frac{m^2}{z^2}\right)\Phi = 0, \quad m^2 = m_0^2 + \left(\xi - \frac{d-1}{4d}R\right).$$

- P_η includes a **potential singular at $z = 0$.**
- P_η is **not** normally hyperbolic.
- Constructing solutions requires **boundary conditions** at $z = 0$.

Does P_η admit fundamental solutions?

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Brute Force construction

We start with a mode decomposition:

- Let $\underline{x} = (t, x_1, ..., x_{d-1})$ and $\underline{k} = (\omega, k_1, ..., k_{d-1})$. Then

$$\Phi(\underline{x}, z) = \int_{\mathbb{R}^d} \frac{d^d \underline{k}}{(2\pi)^{\frac{d}{2}}} e^{i\underline{k}\cdot\underline{x}} \widehat{\Phi}_{\underline{k}}(z)$$

yields $(\lambda \doteq \omega^2 - \sum_{i=1}^{d-1} k_i^2)$

$$P_\eta \Phi = 0 \Longleftrightarrow L\widehat{\Phi}_{\underline{k}} = \left(-\frac{d^2}{dz^2} + \frac{m^2}{z^2} - \lambda\right) \widehat{\Phi}_{\underline{k}} = 0.$$

This is a **singular Sturm-Liouville equation on** $(0, \infty)$**.**

The Endpoint Classification - I

The most general solution of $L\widehat{\Phi}_{\underline{k}} = \lambda\widehat{\Phi}_{\underline{k}}$ is for $\lambda > 0$

$$\widehat{\Phi}_{\underline{k}}(z) = a(\underline{k})\sqrt{z}J_\nu(\sqrt{\lambda}z) + b(\underline{k})\sqrt{z}Y_\nu(\sqrt{\lambda}z),$$

where $\nu = \frac{1}{2}\sqrt{1+4m^2} \geq 0....$ $m^2 \in [-\frac{1}{4}, \infty)$, the BF bound.

Which boundary conditions are allowed? How do we implement them?

- Observe that $\sqrt{z}J_\nu(\sqrt{\lambda}z) \propto_{z\to 0} z^{\nu+\frac{1}{2}}$ and $\sqrt{z}Y_\nu(\sqrt{\lambda}z) \propto_{z\to 0} z^{-\nu+\frac{1}{2}}$

Not obvious how to impose standard (Robin) boundary conditions at $z = 0$!

Singular Sturm-Liouville theory is the answer

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The Endpoint Classification - II

Allowed boundary conditions depend on the solutions near the endpoints:

Endpoint $z = 0$: For an operator $L_V \doteq -\frac{d^2}{dz^2} + V(z)$ we call $z = 0$

(i) *regular (R)* if $\exists z_0 \in (0, \infty)$ such that $V(z) \in L^1(0, z_0)$,

(ii) *limit circle (LC)* if $\exists z_0$ and $\lambda \in \mathbb{C}$ such that all elements in the kernel of the operator $L_V - \lambda$ lie in $L^2(0, z_0)$,

(iii) *limit point (LP)* if it is neither (R) nor (LC).

Endpoint $z \neq 0$: For an operator $L_V \doteq -\frac{d^2}{dz^2} + V(z)$ we call the endpoint

(a) *singular* if $z \to \infty$,

(b) *limit circle (LC)* if $\exists \lambda \in \mathbb{C}$ and $z_0 \in (0, \infty)$ such that all elements in the kernel of the operator $L_V - \lambda$ lie in $L^2(z_0, \infty)$,

(c) *limit point (LP)* if it is not (LC).

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Boundary Conditions - I

Goal: Implement boundary conditions for non regular endpoints

Let $L_V \doteq -\frac{d^2}{dz^2} + V(z)$,

1. Choose $\Phi_1(z)$ the *principal solution* at $z = 0$, *i.e.* the unique one (up to scalar multiples)

$$\lim_{z\to 0} \frac{\Phi_1(z)}{\Phi(z)} = 0 \quad \forall \Phi(z) \mid L_V\Phi = \lambda\Phi,\ \lambda \in \mathbb{C}.$$

2. Pick a *second L^2-solution* $\Phi_2(z)$, linearly indep. from Φ_1 (non unique)
3. Observe that, up to a scalar multiple, if $(L_V - \lambda)\Phi = 0$, $\exists \alpha \in [0, \pi)$ such that

$$\Phi(z) = \cos\alpha\, \Phi_1(z) + \sin\alpha\, \Phi_2(z),$$

and that, for a regular endpoint at $z = 0$,

$$\cos\alpha\, \Phi(0) + \sin\alpha\, \Phi'(0) = 0 \iff \cos\alpha\, W_z[\Phi, \Phi_1] + \sin\alpha\, W_z[\Phi, \Phi_2] = 0$$

where $W_z[\Phi, \Phi_i] = \Phi(z)\Phi_i'(z) - \Phi'(z)\Phi_i(z)$ is the Wronskian.

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Boundary Conditions - II

Key Observations: If we consider solutions which are $L^2(0, z_0)$, $z_0 \in (0, \infty)$

- $\Phi_1(z)$ always exists while $\Phi_2(z)$ can be found if $z = 0$ is R or LC
- The identity with the Wronskian is always meaningful and we call

1. $\alpha = 0$ (generalized) Dirichlet boundary condition,
2. $\alpha = \frac{\pi}{2}$ (generalized) Neumann boundary condition,
3. $\alpha \in (0, \pi)$, $\alpha \neq \frac{\pi}{2}$ (generalized) Robin boundary condition

Boundary Conditions - III

Recall that we consider $L = -\frac{d^2}{dz^2} + \frac{m^2}{z^2}$ and $\lambda = q^2 \doteq \omega^2 - \sum_{i=1}^{d-1} k_i^2$.

The fundamental pair of solutions $(\widehat{\Phi}^1_{\underline{k}}, \widehat{\Phi}^2_{\underline{k}})$ is

$$\widehat{\Phi}^1_{\underline{k}}(z) = \sqrt{\frac{\pi}{2}}\, q^{-\nu} \sqrt{z}\, J_\nu(qz)\,,$$

$$\widehat{\Phi}^2_{\underline{k}}(z) = \begin{cases} -\sqrt{\frac{\pi}{2}}\, q^{\nu} \sqrt{z}\, J_{-\nu}(qz)\,, & \nu \in (0,1)\,, \\ -\sqrt{\frac{\pi}{2}} \sqrt{z} \left[Y_0(qz) - \frac{2}{\pi} \log(q) \right]\,, & \nu = 0\,. \end{cases}$$

$\nu = \frac{1}{2}\sqrt{1+4m^2}$	Classification of $z=0$	Boundary condition at $z=0$
$\nu = \frac{1}{2}$	Regular (R)	$\cot(\alpha)\, \widehat{\Phi}_{\underline{k}}(0) + \widehat{\Phi}'_{\underline{k}}(0) = 0$
$\nu \in [0,1),\ \nu \neq \frac{1}{2}$	Limit-circle (LC)	$-\cot(\alpha)\, W_z\left[\widehat{\Phi}_{\underline{k}}, \widehat{\Phi}^1_{\underline{k}}\right] + W_z\left[\widehat{\Phi}_{\underline{k}}, \widehat{\Phi}^2_{\underline{k}}\right] = 0$
$\nu \in [1,\infty)$	Limit-point (LP)	Not required

Ground States - Mode Expansion I

Goal: Kill two birds with one stone

We construct directly the two-point function of a *ground state*

1. Let $\omega_{2,\mathbb{H}} \doteq (zz')^{\frac{1-d}{2}}\omega_2 \in \mathcal{D}'(\mathring{\mathbb{H}}^{d+1} \times \mathring{\mathbb{H}}^{d+1})$. It holds

$$(P_\eta \otimes \mathbb{I})G_{\mathbb{H}} = (\mathbb{I} \otimes P_\eta)G_{\mathbb{H}} = 0.$$

2. Consider the Fourier transform along $\mathbb{R}^d \ni \underline{x}$. Integral kernel:

$$\omega_{2,\mathbb{H}}(x,x') = \lim_{\epsilon\to 0^+} \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} e^{i\omega(t-t'-i\epsilon)} \int_0^\infty dk \left(\frac{k}{r}\right)^{\frac{d-3}{2}} J_{\frac{d-3}{2}}(kr)\widehat{\omega}_{2,\underline{k}}(z,z').$$

with

$$(L\otimes\mathbb{I})\widehat{\omega}_{2,\underline{k}} = (\mathbb{I}\otimes L)\widehat{\omega}_{2,\underline{k}} = \lambda\widehat{\omega}_{2,\underline{k}}, \quad L = -\frac{d^2}{dz^2} + \frac{m^2}{z^2}.$$

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The case with $\nu \geq 1$

In order to construct $\widehat{\omega}_{2,\underline{k}}$, we need that $iG(x,x') \doteq \omega_{2,\mathbb{H}}(x,x') - \omega_{2,\mathbb{H}}(x',x)$,

$$G(x,x')|_{t=t'} = 0, \quad \partial_t G(x,x') = -\partial_{t'} G(x,x')|_{t=t'} = \delta(x,x').$$

• Finding $\widehat{\omega}_{2,\underline{k}}$ is a problem of eigenfunction expansion of the δ-distribution (*cf.* Titchmarsh 1962)

• $\widehat{\omega}_{2,\underline{k}}$ is not unique and it depends on the boundary conditions.

For $\nu \geq 1$ the choice is **unique** – Fourier-Bessel expansion

$$\widehat{\omega}_{2,\underline{k}} = \sqrt{zz'} \int_0^\infty dq\, q J_\nu(qz) J_\nu(qz'), \quad q^2 = \omega^2 - \sum_{i=1}^{d-1} k_i^2.$$

The outcome is:

$$G(x,x') =$$

$$\lim_{\epsilon \to 0^+} \sqrt{2zz'} \int_0^\infty dq\, q \frac{\sin(\sqrt{k^2+q^2}(t-t'-i\epsilon))}{\sqrt{\pi(k^2+q^2)}} \int_0^\infty dk \left(\frac{k}{r}\right)^{\frac{d-3}{2}} J_{\frac{d-3}{2}}(kr) J_\nu(qz) J_\nu(qz').$$

The case with $\nu \geq 1$

In order to construct $\widehat{\omega}_{2,\underline{k}}$, we need that $iG(x,x') \doteq \omega_{2,\mathbb{H}}(x,x') - \omega_{2,\mathbb{H}}(x',x)$,

$$G(x,x')|_{t=t'} = 0, \quad \partial_t G(x,x') = -\partial_{t'} G(x,x')|_{t=t'} = \delta(x,x').$$

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The case with $\nu \in (0,1)$

In this regime, Robin-like boundary conditions can be imposed

- There exists two known regimes: $c = \cot\alpha \geq 0$ and $c = \cot\alpha < 0$

$$\delta(z-z') = \sqrt{zz'} \int_0^\infty dq\, q \frac{[cJ_\nu(qz) - q^{2\nu} J_{-\nu}(qz)][cJ_\nu(qz') - q^{2\nu} J_{-\nu}(qz')]}{c^2 - 2cq^{2\nu}\cos(\pi\nu) + q^{4\nu}}, \quad c \leq 0$$

or

$$\delta(z-z') = \sqrt{zz'} \int_0^\infty dq\, q \frac{[cJ_\nu(qz) - q^{2\nu} J_{-\nu}(qz)][cJ_\nu(qz') - q^{2\nu} J_{-\nu}(qz')]}{c^2 - 2cq^{2\nu}\cos(\pi\nu) + q^{4\nu}} +$$

$$+2\sqrt{zz'}\, c^{\frac{1}{\nu}} \frac{\sin(\pi\nu)}{\pi\nu} K_\nu(c^{\frac{1}{2\nu}} z) K_\nu(c^{\frac{1}{2\nu}} z'), \quad c > 0$$

For certain Robin boundary conditions, there are bound states!

Ground States - Mode Expansion II

What have we learned?

- We can construct Green operators for all values of m^2 and for all *Robin* boundary conditions, when existent
- In certain regimes there are bound states (bad for quantization)
- The result is consistent with the work of Wald & Ishibashi (CMP 2003)

What is missing?

Ground States - Mode Expansion II

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- The result is consistent with the work of Wald & Ishibashi (CMP 2003)

What is missing?

From $G_{\mathbb{H}}$ to a ground state $\omega_{2,\mathbb{H}}$

From the form of $G_{\mathbb{H}}$ we can construct

$$\omega_{2,\mathbb{H}}(x,x') = \lim_{\epsilon\to 0^+} \int_0^\infty \frac{d\omega}{\sqrt{2\pi}} e^{i\omega(t-t'-i\epsilon)} \int_0^\infty dk \left(\frac{k}{r}\right)^{\frac{d-3}{2}} J_{\frac{d-3}{2}}(kr)\widehat{\omega}_{2,\underline{k}}(z,z') =$$

$$= \lim_{\epsilon\to 0^+} \int_0^\infty dq\, q \frac{e^{i(\sqrt{k^2+q^2}(t-t'-i\epsilon))}}{\sqrt{2\pi(k^2+q^2)}} \int_0^\infty dk \left(\frac{k}{r}\right)^{\frac{d-3}{2}} J_{\frac{d-3}{2}}(kr)\widehat{\omega}_{2,\underline{k}}(z,z').$$

One can prove that

1. $\omega_{2,\mathbb{H}}(x,x')$ is the integral kernel of a bi-distribution $\omega_{2,\mathbb{H}}$ in $\mathcal{D}'(\mathring{\mathbb{H}}^{d+1} \times \mathring{\mathbb{H}}^{d+1})$,
2. $(P_{\eta} \otimes \mathbb{I})\omega_{2,\mathbb{H}} = (\mathbb{I} \otimes P_{\eta})\omega_{2,\mathbb{H}} = 0$ and $\omega_{2,\mathbb{H}}(f,f) \geq 0$ for all $f \in C_0^\infty(\mathring{\mathbb{H}}^{d+1})$,
3. $\omega_{2,\mathbb{H}}$ is maximally symmetric.

$\omega_{2,\mathbb{H}}$ is the ground state

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$\omega_{2,\mathbb{H}}$ is the ground state

Explicit expression for $\omega_{2,\mathbb{H}}$ - I

We can write $\omega_{2,\mathbb{H}}$ in terms of special functions:

- Call $u(x,x') = \cosh^2\left(\frac{\sqrt{2\sigma(x,x')}}{2}\right)$. Then $u = 1 + \frac{\sigma_{\mathbb{M}}(x,x')}{2zz'}$ where $\sigma_{\mathbb{M}}$ is the Minkowski geodesic distance.

Proposition [First Case]

Let $\nu \geq 1$ and let

$$\omega_2^D(x,x') = \lim_{\epsilon\to 0} u_\epsilon^{-\frac{d}{2}-\nu} \frac{F(\frac{d}{2}+\nu, \frac{1}{2}+\nu, 1+2\nu; u_\epsilon^{-1})}{\Gamma(1+2\nu)}$$

The integral kernel of the ground state reads:

$$\omega_{2,\mathbb{H}}(x,x') = \mathcal{N}(\nu,d)\omega_2^D(x,x'),$$

where $\mathcal{N}(\nu,d)$ is a normalization constant.

Explicit expression for $\omega_{2,\mathbb{H}}$ - II

Proposition [Second Case]

Let $\nu \in (0,1)$, for every $\alpha \in [0, \frac{\pi}{2}]$ there exists a ground state built out of $\omega_{2,\mathbb{H}}^{(\alpha)}$ and

$$\omega_2^D(x,x') = \lim_{\epsilon \to 0} u_\epsilon^{-\frac{d}{2}-\nu} \frac{F(\frac{d}{2}+\nu, \frac{1}{2}+\nu, 1+2\nu; u_\epsilon^{-1})}{\Gamma(1+2\nu)}$$

$$\omega_2^N(x,x') = \lim_{\epsilon \to 0} u_\epsilon^{-\frac{d}{2}+\nu} \frac{F(\frac{d}{2}-\nu, \frac{1}{2}-\nu, 1-2\nu; u_\epsilon^{-1})}{\Gamma(1-2\nu)}$$

Notice that

- There is **no ground state** neither for Robin boundary conditions with $\alpha \in (\frac{\pi}{2}, \pi)$ nor for $\nu = 0$ due to the *bound states*, though a causal propagator exists.

Singular Structure of $\omega^{(\alpha)}_{2,\mathbb{H}}$ – Global Form

Theorem

Let $\omega^{(\alpha)}_{2,\mathbb{H}}$ be the ground state for a generic, admissible boundary condition. Then

$$WF(\omega^{(\alpha)}_{2,\mathbb{H}}) = \{(x, k_x, x', k_{x'}) \in T^*(\mathring{\mathbb{H}}^{d+1})^{x2} \backslash \{0\} \,|\, (x_\pm, k_{x_\pm}) \sim (x', k_{x'}),\ k_x \triangleright 0\}$$

where $\sim$ entails that $x_\pm = (\underline{x}, \pm z)$ and $x' = (\underline{x}', z')$ are connected by a lightlike geodesic γ in $\mathbb{M}^{d+1}$, while $k_{x_\pm} = (k_{\underline{x}}, \pm k_z)$ is parallel transported along γ to $k_{x'}$.

Observe that:

- As expected, singularities are reflected at the boundary.
- In every globally hyperbolic subregion of PAdS_{d+1} or, equivalently, of $\mathring{\mathbb{H}}^{d+1}$, the WF is of Hadamard type.

What about BTZ?

Consider $\phi : BTZ \to \mathbb{R}$

$$P\phi = (\Box_{BTZ} - m_0^2 - \xi R)\phi = 0$$

Since $\partial_t, \partial_\varphi$ are Killing fields

$$\phi(t, r, \varphi) = \frac{1}{2\pi} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}} d\omega \, e^{-i\omega t + ik\varphi} \Psi_{\omega k}(r)$$

where the remaining unknown satisfies

$$L\Psi_{\omega k}(z) = \frac{d}{dz}\left(z \frac{d\Psi_{\omega k}}{dz}\right) + q(z)\Psi_{\omega k}(z) = 0,$$

being $z = \frac{r^2 - r_+^2}{r^2 - r_-^2} \in (0, 1)$ and setting $\mu^2 = m_0^2 \ell^2 - 6\xi$

$$q(z) = \frac{1}{4(1-z)} \left[\frac{\ell^2(\omega \ell r_+ - k r_-)^2}{(r_+^2 - r_-^2)z} - \frac{\ell^2(\omega \ell r_- - k r_+)^2}{(r_+^2 - r_-^2)} - \frac{\mu^2}{1-z} \right],$$

The solutions

Using Froebenius method one can construct two linearly independent solutions ($\mu^2 \neq (n-1)^2 - 1,\ n \in \mathbb{N}$)

$$\begin{cases} \Psi_1(z) = z^\gamma (1-z)^\beta F(a, b, a+b-c; 1-z) \\ \Psi_2(z) = z^\gamma (1-z)^{1-\beta} F(c-a, c-b, c-a-b+1; 1-z) \end{cases}$$

where

$$\gamma = -i \frac{\ell^2 r_+ (\omega - k\Omega_{\mathcal{H}})}{2(r_+^2 - r_-^2)} \quad \text{and} \quad \beta = \frac{1}{2}\left(1 + \sqrt{1+\mu^2}\right),$$

and

$$\begin{cases} a = \frac{1}{2}\left(1 + \sqrt{1+\mu^2} - i\ell \frac{\ell(\omega - k\Omega_{\mathcal{H}})}{r_+ - r_-} + i\ell \frac{k}{r_+}\right) \\ b = \frac{1}{2}\left(1 + \sqrt{1+\mu^2} - i\ell \frac{\ell(\omega - k\Omega_{\mathcal{H}})}{r_+ + r_-} + i\ell \frac{k}{r_+}\right) \\ c = 1 - i \frac{\ell^2 r_+ (\omega - k\Omega_{\mathcal{H}})}{r_+^2 - r_-^2} \end{cases}$$

Endpoint Classification

We need to control the square integrability of the solutions at

$$z = 0 \text{ (horizon) and } z = 1 \text{ (conformal infinity)}$$

- $z = 0$ is always **Limit-point**
- $z = 1$ can be either **Limit-point** or **Limit-circle**

$\mu^2 = m_0^2\ell^2 - 6\xi$	Classification of $z = 1$	Boundary condition at $z = 1$
$\mu^2 \in (-1, 0)$,	Limit-circle (LC)	$\cot(\alpha)\, W_z\left[\Psi_{\omega k}, \Psi_1\right] + W_z\left[\Psi_{\omega k}, \Psi_2\right] = 0$
$\mu^2 \geq 0$	Limit-point (LP)	Not required

Construction of the ground state - I

Having under control the boundary conditions we can

construct the ground state for the KG field

Differences from *PAdS*:

- we need to consider positive frequencies with respect to $\partial_t + \Omega_{\mathcal{H}} \partial_\varphi$, *i.e.* $\widetilde{\omega} = \omega - \Omega_{\mathcal{H}} k$
- we are no longer dealing with an eigenvalue problem but with a *quadratic operator pencil*

Repeating the same procedure as in $PAdS_{d+1}$ we obtain *3 cases*

1. $\mu^2 \geq 0$, no boundary condition required:

$$\omega_2(x, x') = \lim_{\epsilon \to 0^+} \sum_{k \in \mathbb{Z}} e^{ik(\varphi - \varphi')} \int_0^\infty \frac{d\widetilde{\omega}}{(2\pi)^2} e^{i\widetilde{\omega}(t - t' - i\epsilon)} \left(\frac{A}{B} - \frac{\bar{A}}{\bar{B}} \right) C \Psi_1(z) \Psi_1(z')$$

with $A = \frac{\Gamma(c-1)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$, $B = \frac{\Gamma(c-1)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}$ and $C = \frac{\ell^4}{4(r_+^2 - r_-^2)\sqrt{1+\mu^2}}$.

Construction of the ground state - II

2. $-1 < \mu^2 < 0$, $\alpha \in (0, \alpha_*)$ and $\alpha_* = \arctan\left(\frac{\Gamma(2\beta-1)\left|\Gamma(1-\beta+i\frac{\ell k}{r_+})\right|^2}{\Gamma(1-2\beta)\left|\Gamma(\beta+i\frac{\ell k}{r_+})\right|^2}\right)$

$$\omega_2(x,x') =$$

$$\lim_{\epsilon\to 0^+}\sum_{k\in\mathbb{Z}} e^{ik(\varphi-\varphi')}\int_0^\infty \frac{d\widetilde{\omega}}{(2\pi)^2} e^{i\widetilde{\omega}(t-t'-i\epsilon)}\left(\frac{A\bar{B}-\bar{A}B}{|\cos\alpha B-\sin\alpha A|^2}\right) C\Psi_1(z)\Psi_1(z')$$

3. $-1 < \mu^2 < 0$, $\alpha \in (\alpha_*, \pi)$ bound states $\Longrightarrow$ **no ground state**.

- The ground states are *locally* of Hadamard form.

Towards an existence theorem - I

Goal: Can we formulate an existence and "uniqueness" theorem?

Consider a *standard static* spacetime $N = \mathbb{R} \times M$ with a *timelike boundary*:

1. (N, h) and $(\partial N, \iota^* h)$ are Lorentzian manifolds with $\partial N = \mathbb{R} \times \partial M$

$$h = -\beta dt^2 + g \quad \beta \in C^\infty(M; (0, \infty))$$

2. (M, g) is a **Riemannian manifold with boundary and of bounded geometry** [Schick '01 & Amman, Große, Schneider '16]
   - there exists $(\widehat{M}, \widehat{g})$ such that $\dim \widehat{M} = \dim M$ and

$$M \subset \widehat{M} \quad \widehat{g}|_M = g,$$

   - $(\widehat{M}, \widehat{g})$ is of bounded geometry, *i.e.*,

$$r_{inj}(\widehat{M}) > 0, \quad \|\nabla^k \widehat{R}\|_{L^\infty(\widehat{M})} < \infty \; \forall k \in \mathbb{N} \cup \{0\},$$

   - $(\partial M, \iota_M^* \widehat{g})$ is of bounded geometry.

Towards an existence theorem - II

Let (M, g) be a manifold with boundary and of bounded geometry

1. consider *a geodesic atlas* $(U_\beta^{geo}, \kappa_\beta^{geo})$, $\beta \in J$, J being an index set
2. consider $\{h_\beta^{geo}\}_{\beta \in J}$ a partition of unity subordinated to the geodesic atlas and let $H^{s,geo}(\widehat{M})$ be the set of $u \in \mathcal{D}'(\widehat{M})$

$$\|u\|^2_{H^{2,geo}(\widehat{M})} = \sum_{\beta \in J} \|(h_\beta^{geo} u)| \circ \kappa_\beta^{geo}\|^2_{H^2(\mathbb{R}^m)} < \infty$$

3. $H^{s,geo}(\widehat{M})$ is isomorphic to $W^s(\widehat{M})$ the completion of

$$\mathcal{E}^s(\widehat{M}) = \{f \in \mathcal{E}(\widehat{M}) \mid f, \nabla f \ldots \nabla^s f \in L^2(\widehat{M})\} \quad \|f\|^2 = \sum_{i=0}^{s} \|\nabla^i f\|_{L^2(\widehat{M})}$$

Theorem (Große & Schneider '13)

Let (M, g) be a manifold with boundary and of bounded geometry and let

$$H^s(M) = \{[u] \mid u \in H^s(\widehat{M}) \text{ and } u \sim u' \text{ iff } (u - u')|_M = 0\}$$

There exists a **continuous surjective** *map* $\Gamma : H^s(M) \to H^{s-\frac{1}{2}}(\partial M)$.

The main problem

Let (N, h) be a standard static spacetime with timelike boundary and

$$\Phi : N \to \mathbb{R} \quad \text{such that} \quad \Box_h \Phi = 0,$$

Assume (for simplicity) h to be *ultrastatic, i.e.* $\beta = 1$.

Question: Which are the fundamental solutions for $\Box_h$?

We look for $G \in \mathcal{D}'(\mathring{N} \times \mathring{N})$, $\mathring{N} = N \setminus \partial N$

$$\begin{cases} (\Box_h \otimes \mathbb{I})G = (\mathbb{I} \otimes \Box_h)G = 0 \\ G|_{t=t'} = 0 \text{ and } \partial_t G = -\partial_{t'} G = \delta_M \end{cases}$$

Since $\Box_h = -\partial_t^2 + \Delta_g$,

Answer: Characterize the self-adjoint extensions of Δ_g

Boundary Triples

Let $S : D(S) \subset \mathsf{H} \to \mathsf{H}$ be a closed symmetric operator.

Definition

A **boundary triple** for S^* is a triple $(\mathsf{h}, \gamma_0, \gamma_1)$ where $\gamma_i : D(S^*) \to \mathsf{h}$,

$$(S^* f, f')_{\mathsf{H}} - (f, S^* f')_{\mathsf{H}} = (\gamma_1 f, \gamma_0 f')_{\mathsf{h}} - (\gamma_0 f, \gamma_1 f')_{\mathsf{h}},$$

and the map $\gamma : D(S^*) \to \mathsf{h} \times \mathsf{h}$, $f \mapsto \gamma(f) = (\gamma_0(f), \gamma_1(f))$ is surjective.

Theorem (Grubb '68 & Malamud '92)

Let S be as above and let $\mathcal{N}_\pm = \ker(S^* \pm i\mathbb{I})$. If $\dim \mathcal{N}_+ = \dim \mathcal{N}_-$, then a boundary triple $(\mathsf{h}, \gamma_0, \gamma_1)$ exists. Then, to any self-adjoint operator $\Theta : D(\Theta) \to \mathsf{h}$, it corresponds a self-adjoint extension of S:

$$S_\Theta = S^*|_{\ker(\gamma_1 - \Theta\gamma_0)}$$

Every self-adjoint extension of S is of this form.

Application to Δ_g

Let (M, g) be a Riemannian manifold with boundary and of bounded geometry and Let Δ_g be the Laplace-Beltrami operator (uniformly elliptic) with

$$D_{max}(\Delta_g^*) = \{f \in L^2(M) \mid \Delta_g f \in L^2(M)\} \simeq H^2(M).$$

Theorem (Grubb '68)

Let $\Gamma_0 \equiv \Gamma : H^2(M) \to H^{\frac{3}{2}}(M)$ be the Lions trace and let $\Gamma_1 = -\Gamma\nabla_n : H^2(M) \to H^{\frac{1}{2}}(M)$. Then

$$(L^2(\partial M), \gamma_0, \gamma_1)$$

is a boundary triple for Δ_g^* if $\gamma_0 = \iota_+\Gamma_0$ and $\gamma_1 = j_+\Gamma_1$ where

$$\iota_+ : H^{\frac{3}{2}}(\partial M) \to L^2(\partial M) \quad and \quad j_+ : H^{\frac{1}{2}}(\partial M) \to L^2(\partial M).$$

The propagator(s) for $\Box_h$ - I

Assume that

1. (N, h) is a static Lorentzian spacetime with timelike boundary
2. $(L^2(\partial M), \gamma_0, \gamma_1)$ is the boundary triple associated to Δ_g^*
3. Θ is a densely defined self-adjoint operator on $L^2(\partial M)$ such that $\Delta_\Theta \doteq \Delta_g^*|_{D(\Delta_\Theta)}$, where $D(\Delta_\Theta) \doteq \ker(\gamma_1 - \Theta\gamma_0)$.
4. the spectrum of Δ_Θ is bounded from below

The propagator(s) for $\Box_h$ - II

Theorem (C.D. & Nicoló Drago)

The advanced and retarded fundamental solutions for $\Box_h$ are completely determined in terms of the bidistributions $\mathcal{G}_\Theta^- = \theta(t-t')\mathcal{G}_\Theta$ and $\mathcal{G}_\Theta^+ = -\theta(t'-t)\mathcal{G}_\Theta$, where $\mathcal{G}_\Theta \in \mathcal{D}'(\mathring{N} \times \mathring{N})$ is such that, for all $f \in \mathcal{D}(\mathring{N})$

$$\mathcal{G}_\Theta(f_1, f_2) \doteq \int_{\mathbb{R}^2} dt\,dt' \left(f_1(t) \middle| A_\Theta^{-\frac{1}{2}} \sin\left[A_\Theta^{\frac{1}{2}}(t-t')\right] f_2(t') \right),$$

where $f(t) \in H^2(M)$ denotes the evaluation of f, regarded as an element of $C_c^\infty(\mathbb{R}, H^\infty(M))$ and $A_\Theta^{-\frac{1}{2}} \sin\left[A_\Theta^{\frac{1}{2}}(t-t')\right]$ is defined exploiting the functional calculus for A_Θ. Moreover it holds that

$$\mathsf{G}_\Theta^\pm : \mathcal{D}(\mathring{N}) \to C^\infty(\mathbb{R}, H_\Theta^\infty(M)),$$

where $H_\Theta^\infty(M) \doteq \bigcap_{k \geq 0} D(\Delta_\Theta^k)$. In particular,

$$\gamma_1\left(\mathsf{G}_\Theta^\pm f\right) = \Theta \gamma_0\left(\mathsf{G}_\Theta^\pm f\right) \qquad \forall f \in C_0^\infty(\mathring{N}).$$

And now?

Outlook

- We have constructed the causal propagator and the ground state for any massive scalar field in PAdS_{d+1} and in *BTZ* with arbitrary Robin-type boundary conditions,
- We have developed a general framework to discuss the existence of the fundamental solutions

To do

- Better understand the role of bound states,
- Apply our procedure to asymptotically AdS (static) spacetimes,
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