Renormalization of QFT on Riemannian manifolds

Quantum fields, scattering and spacetime horizons: *mathematical challenges*

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Euclidean QFT.

(M, g) closed compact Riemannian manifold dimension d, volume form $dv(x)$, $-\Delta$ Laplace Beltrami.
Sequence $\sigma = \{0 = \lambda_0 < \lambda_1 \leqslant \cdots \leqslant \lambda_k \to +\infty\}$ and $(e_\lambda)_{\lambda \in \sigma}$ **eigenfunctions**

$$-\Delta e_\lambda = \lambda e_\lambda.$$

Green kernel :

$$\boxed{P(x, y) = \sum_{\lambda \in \sigma, \lambda > 0} \lambda^{-1} e_\lambda(x) e_\lambda(y).}$$

The Gaussian Free Field.

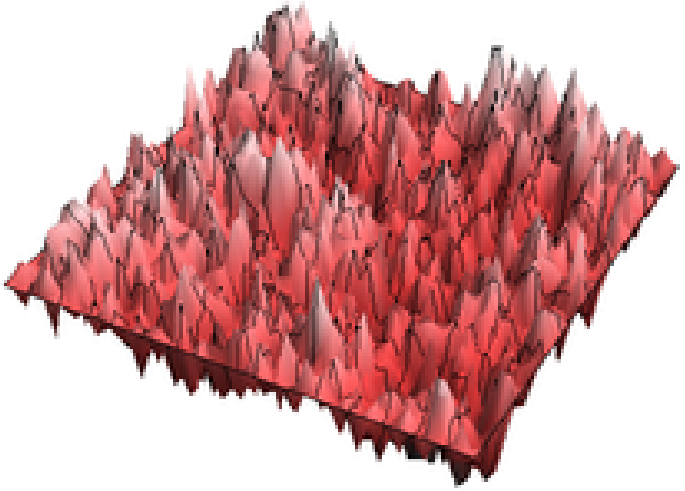
Generalizes Brownian motion. **Explicit realization** of *GFF*. For $(c_\lambda)_\lambda$ gaussian centered i.i.d, random series :

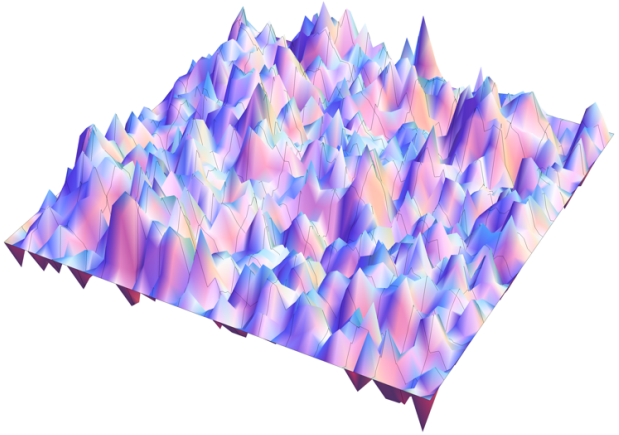
$$\boxed{\phi = \sum_{\lambda \in \sigma \setminus \{0\}} \frac{c_\lambda}{\sqrt{\lambda}} e_\lambda .}$$

Intuitively random Gaussian vector in some ∞ dim space.

DGFF pictures not from me.

Like Brownian motion, obtained as scaling limit.





Random distributions.

ϕ in L^2 ?

$$\begin{aligned}\mathbb{E}\left(\|\phi\|^2_{L^2(M)}\right) &= \mathbb{E}\left(\left\langle \sum \frac{c_\lambda}{\sqrt{\lambda}} e_\lambda, \sum \frac{c_\lambda}{\sqrt{\lambda}} e_\lambda \right\rangle\right) \\ &= \sum_\lambda \frac{1}{\lambda} \underbrace{\mathbb{E}(c_\lambda^2) \langle e_\lambda, e_\lambda \rangle}_{=1} = \sum_\lambda \frac{1}{\lambda}.\end{aligned}$$

Dimension d Weyl's law $\lambda_n \sim Cn^{\frac{2}{d}}$ **divergent** when $d \geqslant 2$.

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Random distribution need to **smear** the field ϕ. For $h \in C^\infty(M)$, $\phi(h)$ Gaussian random variable.
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$$\mathbb{E}\left(\phi(h_1)\phi(h_2)\right) = \int_{M\times M} P(x,y) h_1(x) h_2(y) dv(x) dv(y).$$

P covariance of GFF measure $\int [d\phi] e^{-\frac{1}{2}\langle \phi, (-\Delta)\phi\rangle}$ on $\mathcal{D}'(M)$.

Quantum field theory in curved space–times philosophy.

QFT curved space-times : metric g *classical*, interacting with *scalar quantum field* ϕ. Discuss two problems with toy model : renormalization and covariance.

Toy model and renormalization problem.

External potential $V \in C^\infty(M)$, $V \geqslant 0$, action $S = \int_M \left(\phi(-\Delta)\phi + \underbrace{V(x)\phi^2(x)} \right) dv(x)$ where $V(x)\phi^2(x)$ is an interaction term of ϕ with V.

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$$Z(V) = \int [d\phi] exp \left(-\frac{1}{2} \int_M \phi(-\Delta)\phi + V\phi^2 dv \right)$$

Also when $V =$ constant λ.
Some problems of QFT can be understood with probability language. Since $\phi \in H^{<0}(M)$, ϕ^2 **ill–defined** and surely not random variable valued in $\mathcal{D}'(M)$.
Renormalization problem.

Invariants.

Topological Field Theories

$$\boxed{\text{closed manifold } M \longmapsto \text{ partition function } Z(M) \sim \sum_{n=0}^{\infty} h^n \underbrace{F_n(M)}.}$$

$F_n(M)$ topological invariant independent of C^∞ structure of M.

Flat space : Belkale–Brosnan, Bogner–Weinzierl show Feynman amplitudes are **special numbers** i.e. **periods**.

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Flat space : Belkale–Brosnan, Bogner–Weinzierl show Feynman amplitudes are **special numbers** i.e. **periods**.
On **curved space**, QFT numbers could be anything.
More structure on M, complex, Riemannian, bundles $E \mapsto M$, how to get invariants of corresponding structures ?
But $Z(\lambda)$ **isometry invariant** of metric g i.e. depend only on **Riemannian structure**. What information on (M, g) contained in $Z(\lambda)$?

Work in progress.

Definition (Moduli space of metrics)

$\mathcal{R}(M) = C^\infty$ *Metrics*$/C^\infty$ *Diffeos.* $[g] \in \mathcal{R}(M)$ *called* **isometry class** *of metric.*

Proposition (Observation)

(M, g) *Riemannian surface.*

$$\int_M :\phi^2: dv = \lim_{\varepsilon \to 0} \int_M \phi_\varepsilon^2 - \mathbb{E}\left(\phi_\varepsilon^2\right) dv \tag{1}$$

where $\phi_\varepsilon = e^{\varepsilon\Delta}\phi$. *Then* $\int_M :\phi^2: dv$ *well defined* **random variable** *whose law gives Euler characteristic* $\chi(M)$ *and* **lenght spectrum** *of* M.

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Finite dimensional $N \subset \mathcal{R}(M)$ *of metrics with* $K < -\varepsilon < 0$. $I \subset \mathcal{R}(M)$ *subset of metric class with same given partition function*

$$\boxed{Z(\lambda) = \left\langle \exp\left(-\lambda \int_M :\phi^2:\right)\right\rangle_{g_0}.}$$

Then $I \cap N$ **finite**.

Factorize.

$$\begin{aligned} Z(V) &= \int [d\phi] exp\left(-\frac{1}{2}\int_M \phi(-\Delta)\phi + V\phi^2 dv\right) \\ &= \underbrace{\det(-\Delta)^{-\frac{1}{2}}}_{\text{ill–defined}} \det\left(I + (-\Delta)^{-1}V\right)^{-\frac{1}{2}} \end{aligned}$$

where $\det(-\Delta)^{-\frac{1}{2}} = \underbrace{\left(\prod_{\lambda\in\sigma\setminus\{0\}} \lambda\right)^{-\frac{1}{2}}}_{\textbf{infinite product.}}$.

Zeta regularization QFT (Ray–Singer 1971, Hawking 1975).

Introduce spectral zeta function

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Therefore

Definition

$$\sum_{\lambda \in \sigma, \lambda > 0} \log(\lambda) = -\zeta'(0)$$

well–defined and so is $\det(-\Delta)^{-\frac{1}{2}}$.

Divergences in operator theory.

Dimension $d \geqslant 2$, $(-\Delta)^{-1} V \in \Psi^{-2}(M)$ compact but **not trace class**.

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If $n > \frac{d}{2}$, $((-\Delta)^{-1}V)^n \in \Psi^{-2n}(M)$ **trace class**. Hence

$$\begin{aligned}\log\det(1+(-\Delta)^{-1}V) &= \underbrace{\sum_{1\leqslant n\leqslant \frac{d}{2}} \frac{(-1)^{n+1}}{n} TR\left(\left((-\Delta)^{-1}V\right)^n\right)}_{\text{divergent}} \\ &+ \underbrace{\sum_{n>\frac{d}{2}} \frac{(-1)^{n+1}}{n} TR\left(\left((-\Delta)^{-1}V\right)^n\right)}_{\text{converges when } \|V\|_{L^\infty} \text{ small}}\end{aligned}$$

Traces to Feynman graphs.

Definition (Feynman rules)

G graph.

$$t_G = \prod_{e \in E(G)} \underbrace{P(x_{i(e)}, x_{j(e)})}_{\textit{Green kernel}} \tag{2}$$

where the product is over the edges e in G, $(i(e), j(e))$ vertices connected by edge e.

$$\log\det(1 + (-\Delta)^{-1} V) = \sum_{G_n} \frac{(-1)^{n+1}}{n} \int_{M^n} t_{G_n}(x_1, \ldots, x_n) V(x_1) \ldots V(x_n) dv(x_1) \ldots dv(x_n)$$

where G_n **polygon** with n-vertices.

Local renormalization II.

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$$\underbrace{\sum_{1 \leqslant n \leqslant \frac{d}{2}} \frac{(-1)^{n+1}}{n} TR\left(\left((-\Delta)^{-s} V\right)^n\right)}_{\text{regularize}} + \underbrace{\sum_{n > \frac{d}{2}} \frac{(-1)^{n+1}}{n} TR\left(\left((-\Delta)^{-1} V\right)^n\right)}_{\text{converges when } \|V\|_{L^\infty} \text{ small}}$$

Analytic continuation of $TR\left(\left((-\Delta)^{-s} V\right)^n\right)$.

Local renormalization III : extraction.

When $d = 4$, renormalize only the tadpole and 2-gon by **extracting the singular parts** :

- $$\boxed{TR\left(\Delta^{-s}V\right) = \frac{1}{96\pi^2(s-1)}\int_M R_{\text{scal}}V(x) + O(1)}$$

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- $$TR\left(\Delta^{-s}V\Delta^{-s}V\right) = TR\left(\Delta^{-2s}V^2\right) + TR(\underbrace{\Delta^{-s}V[\Delta^{-s},V]}_{\in\Psi^{-4s-1}(M)})$$

  by **cyclicity of trace**, find :

$$\boxed{TR\left(\Delta^{-s}V\Delta^{-s}V\right) = \frac{1}{32\pi^2}\frac{1}{2s-2}\int_M dxV^2(x) + O(1).}$$

Local renormalization IV : subtraction of **local** counterterms.

When $\dim(M) = 4$, renormalization by **local subtraction of counterterms** :

$$\lim_{s\to 1} \log \left(\int [d\phi] \exp \left(-\frac{1}{2}\int_M \phi(-\Delta)\phi + V(x)\phi^2(x) - \underbrace{\frac{1}{s-1}\left(\frac{R_{scal}V(x)}{96\pi^2} - \frac{V^2(x)}{64\pi^2}\right)}_{\textbf{local counterterm}} \right) \right)$$

has well–defined limit.

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For **analytic continuation of** $TR((A^{-s}V)^n)$ where A elliptic of order 2, results from Kontsevich–Vishik, Lesch, Paycha on ΨDO traces.

Local renormalization V : Epstein–Glaser point of view.

Example

In our case : $\lim_{s\to 1} P_s(x,y)^2 - \frac{1}{64\pi^2(s-1)}\delta(x-y) \in \mathcal{D}'(M^2)$ is a distributional extension of $P^2(x,y) \in C^\infty(M^2 \setminus d_2)$.

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Renormalization in the sense of Epstein–Glaser (1975), Stora (1982, 2013), Brunetti-Fredenhagen (2000), Hollands–Wald (2001), Nikolov–Todorov–Stora (2013)

$$\lim_{s\to 1}\left(t_{G_n}(s) - \underbrace{c(s)}_{\text{counterterm}}\right) \in \mathcal{D}'(M^n)$$

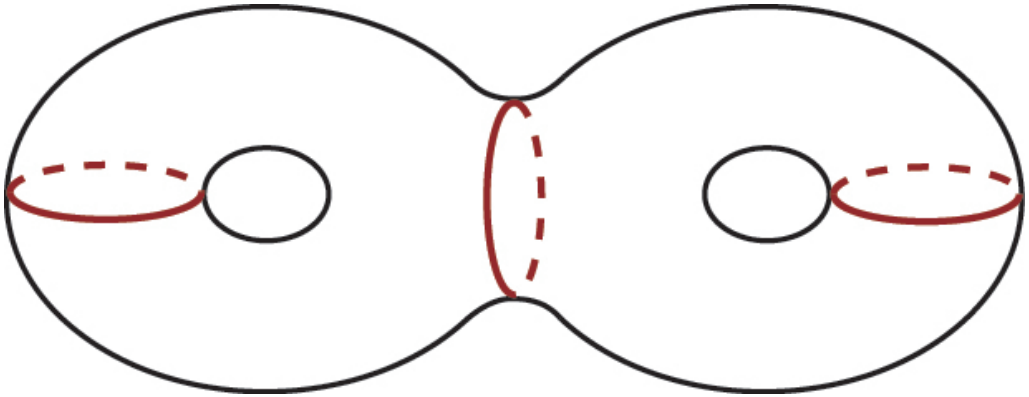
is a **distributional extension** of t_{G_n} whose *WF* over diagonal $d_n = \{(x,\dots,x); x \in M\} \subset M^n$ is contained in conormal of d_n.

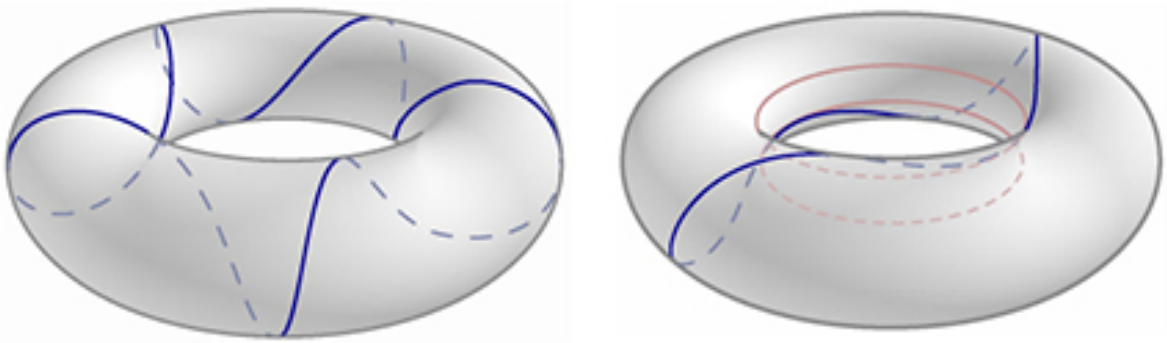
Local renormalization V : Epstein–Glaser point of view.

Recipe : **choose** some distributional extension $\mathcal{R}(t_{G_n})$ for $t_{G_n}, n \leqslant \frac{d}{2}$ whose WF has right property then renormalized determinant $\mathcal{R}\det(1+(-\Delta)^{-1}V)$ defined as :

$$\boxed{Z(V) = \exp\left(\sum_{2\leqslant n\leqslant \frac{d}{2}} \frac{(-1)^{n+1}}{n}\left\langle \mathcal{R}(t_{G_n}), V^{\otimes n}\right\rangle + \sum_{n>\frac{d}{2}} \frac{(-1)^{n+1}\, TR(((-\Delta)^{-1}V)^n)}{n}\right)}$$

$V \longmapsto Z(V)$ **local** functional.





Periods of geodesic flow.

Definition (Periods)

Subset of $\mathbb{R}$

$$\mathcal{P}(g) = \{T \text{ s.t. } \Phi^T(x;\xi) = (x;\xi) \text{ for } (x;\xi) \in S^*M\} \subset \mathbb{R}$$

where $(\Phi^t)_t$ **Hamiltonian flow** *on the unitary cosphere* S^*M *defined by* $g_{\mu\nu}(x)\xi^\mu\xi^\nu$.

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Closed orbit γ in constant energy hypersurface $\{H = 1\} \simeq S^*M \subset T^*M$,

$\boxed{\text{Period}(\gamma) = \int_{\gamma \subset H=1} \theta}$ where $\theta = pdq$ Liouville form.

Consequence.

Let (g_1, g_2) be two metrics. Denote c_n amplitude for n-gon graph. If $c_n(g_1) = c_n(g_2)$ for all $n > \frac{d}{2}$ then $\mathrm{Spec}(M, g_1) = \mathrm{Spec}(M, g_2)$.

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Zeroes of entire function $D(\lambda) = \exp\left(\sum_{n>\frac{d}{2}} \frac{(-1)^{n+1}\lambda^n}{n} c_n(g)\right)$.

$$\boxed{S_{EH}(g) = \mathrm{Res}|_{s=\frac{d}{2}-1}\left(\int_0^\infty t^{s-1} \sum_{z\in\{D=0\}} e^{-tz} dt\right).}$$

$\dim(M) = 2$, $\chi(M)$ by Gauss–Bonnet.

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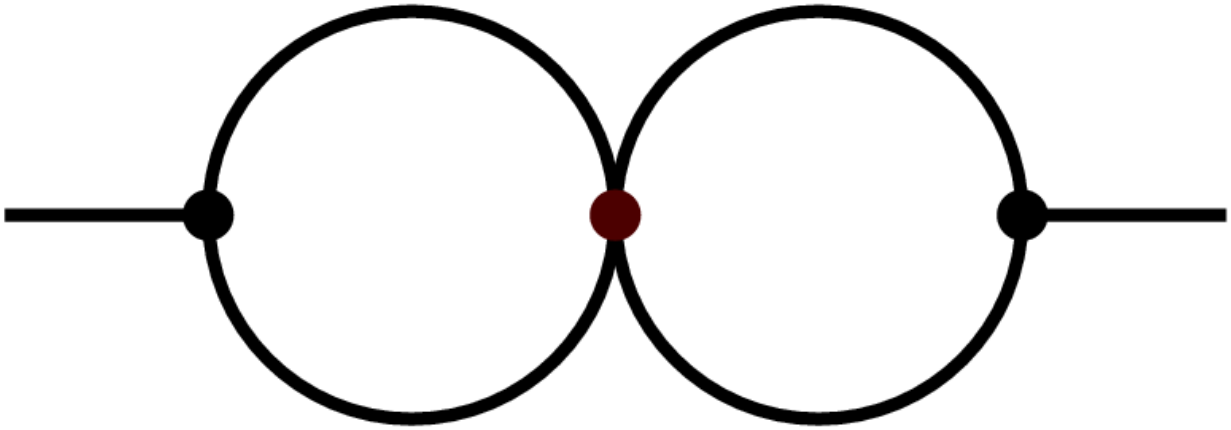
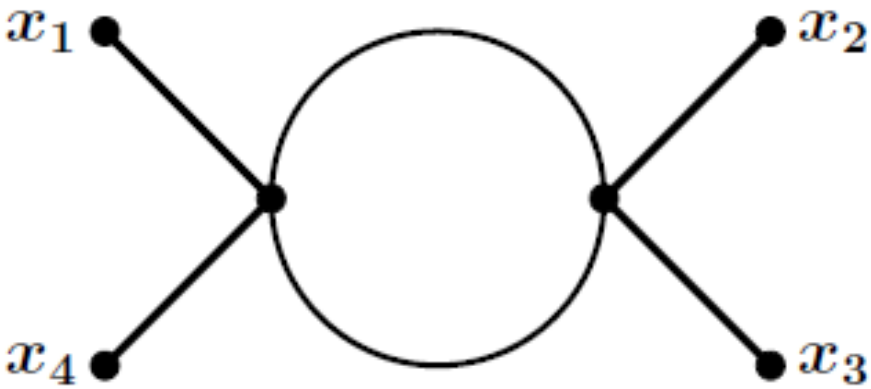
$\dim(M) = 2$, $\chi(M)$ by Gauss–Bonnet.
Moreover by Colin de Verdière, Chazarain and Duistermaat–Guillemin's trace formulas, if (g_1, g_2) have negative curvature then same lenght spectrum $\mathcal{P}(g_1) = \mathcal{P}(g_2)$. Rigidity from Guillemin–Kazhdan and compactness of isospectral metrics from Osgood-Philips-Sarnak should give finiteness.

Diagrams of ϕ^4 theory

Need method to go beyond free theory.





The idea of Eugene Speer.

Example

Triangle graph. On $\mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4$, $G(x,y) = \frac{c}{\|x-y\|^2}$, the amplitude :

$$\frac{c}{\|x_1 - x_2\|^2} \frac{c}{\|x_2 - x_3\|^2} \frac{c}{\|x_3 - x_1\|^2}$$

becomes after regularization

$$\frac{c}{\|x_1 - x_2\|^{2s_{12}}} \frac{c}{\|x_2 - x_3\|^{2s_{23}}} \frac{c}{\|x_3 - x_1\|^{2s_{31}}}$$

where $(s_{12}, s_{23}, s_{31}) \in \mathbb{C}^3$ are **complex parameters**.
Number of complex parameters=number of edges in G

Multiple zeta regularization.

Definition (Multiple zeta regularization)

G a graph with n vertices then regularized Feynman amplitudes :

$$t_G((s_e)_{e\in E(G)}) = \prod_{e\in E(G)} \underbrace{P_{s_e}(x_{i(e)}, x_{j(e)})}_{\text{Green kernel of } (-\Delta)^{-s}} \tag{3}$$

which is in $C^\infty(M^n \setminus \text{diagonals})$ depending on parameters $(s_e)_{e\in E(G)}$.

Related to Duetsch–Fredenhagen–Keller–Rejzner, Guéré–Hack–Pinamonti.

Meromorphic germs with linear poles.

Example

On $\mathbb{C}^2$, $P \in \mathbb{C}[X_1, X_2]$:

$$\frac{P(z_1, z_2)}{(z_1 + z_2)^2 z_1 z_2}$$

Singular on $\{z_1 = 0\}, \{z_2 = 0\}, \{z_1 + z_2 = 0\}$.

Theorem (Berline–Vergne(2015), Guo–Paycha–Zhang(2015))

Direct sum decomposition :

$$\textit{polar germs} \oplus \textit{holomorphic part}.$$

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$$\frac{z_1 + 2z_2}{z_1(z_1 + z_2)z_2} = \frac{2}{z_1(z_1 + z_2)} + \frac{1}{(z_1 + z_2)z_2}$$

With Bin Zhang using Guo–Paycha–Zhang.

Bin Zhang

Main result.

Theorem (D-Zhang)

G oriented graph with n vertices, $\forall \varphi \in C^\infty(M^n)$,

$$\int_{M^n} t_G(s)\varphi \underbrace{d^d x_1 \dots d^d x_n}_{\textit{Riemannian vol.}} \tag{4}$$

extends to a meromorphic germ of $(s_e)_{e\in E(G)}$ with linear poles at $(s_e = 1)_{e\in E(G)}$.

Pole structure : simplified version.

Definition (Divergent graphs.)

Graph G divergent if there is some subgraph G_i such that $\frac{d}{2} b_1(G_i) - e(G_i) \in \mathbb{N}$ where $e(G_i)$ number of edges in G_i.

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G oriented graph with n vertices, $\forall \varphi \in C^\infty(M^n)$, $\langle t_G(s), \varphi \rangle$ belongs to $\mathcal{M}_+$-module generated by $\prod_{G' \subset G} \frac{1}{\sum_{e \in E(G')} s_e - e(G')}$.

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In practice

$$\boxed{\langle t_G(s), \varphi \rangle \in \bigoplus_{F \in \mathrm{Div}(G)} \left(\prod_{G' \in F} \frac{1}{\sum_{e \in E(G')} s_e - e(G')} \right) \mathcal{M}_+ \bigoplus \mathcal{M}_+.}$$

Idea of the proof I.

Relation with **heat kernel** and decomposition

$$\boxed{(-\Delta)^{-s} = \frac{1}{\Gamma(s)} \int_0^1 \left(e^{t\Delta} - \Pi\right) t^s \frac{dt}{t} + \text{holo.}}$$

Heat kernel expansion :

$$\boxed{e^{t\Delta}(x,y) \sim \sum_{k=0}^{\infty} \psi(d^2) \frac{e^{-\frac{d^2(x,y)}{4t}}}{(4\pi t)^{\frac{d}{2}}} a_k(x,y) t^k}$$

Idea of the proof II.

$E(G) = E$, distance function $d : M \times M \mapsto \mathbb{R}_{\geq}$. Analytic continuation in $(s_e)_{e=1}^{E}$ of

$$\int_{[0,1]^E} dt_1 \dots dt_E \, t_1^{s_1 - 1} \dots t_E^{s_E - 1} \underbrace{\exp\left(- \sum_{e=1}^{E} \frac{d^2(x_{i(e)}, x_{j(e)})}{4 t_e} \right)}_{\textbf{non smooth}} \dots$$

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Work in space $M^{V(G)} \times \underbrace{[0,1]^E}$.

Idea of the proof III : blow-up.

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s.t. the pull–back $\pi^* \exp\left(-\sum_{e=1}^{2} \frac{|x_1 - x_2|^2}{4t_e}\right) \in C^\infty\left([0,1]^2 \times (\mathbb{R}^d)^2\right)$.

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$$\boxed{\exp\left(-\frac{|h|^2(1+u_1)}{4}\right)}$$

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Reduce to integrals of the form

$$\int_{[0,1]^2} du_1 du_2 \underbrace{\left(u_1^{s_1-1} u_2^{s_1+s_2-\frac{d}{2}-1}\right)}_{\text{gives linear poles}} \underbrace{\chi(u_1, u_2)}_{C^\infty}$$

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Inspired by Speer (1969) and Hepp (1966), blow-ups are encoded combinatorially by spanning trees of the graph G which correspond to **simplices** (called Hepp sector) of $[0,1]^E$.
Draw picture.

Thanks for your attention !