Decay of the local energy and scattering theory for the charged Klein-Gordon equation in the exterior De Sitter-Reissner-Nordström spacetime

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The exterior De Sitter-Reissner-Nordström (DSRN) spacetime

In Boyer-Lindquist coordinates $(t, r, \omega) \in \mathbb{R}_t \times]0, +\infty[_r \times \mathbb{S}^2_\omega$, the **DSRN metric** g is given by

$$g = F(r)\mathrm{d}t^2 - F(r)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\omega^2, \qquad F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{2r^2} - \frac{\Lambda r^2}{3}$$

where $M > 0$ is the black hole mass, $Q \neq 0$ its electric charge and $\Lambda > 0$ is the cosmological constant.

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Let $A_\mu = (Q/r)\,\mathrm{d}t$. Then (g, A_μ) solves the **Einstein-Maxwell field equation**

$$\mathrm{Ric}_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}, \qquad T_{\mu\nu} = \frac{1}{4}g_{\mu\nu}\mathcal{F}^{\sigma\rho}\mathcal{F}_{\sigma\rho} - \mathcal{F}_{\mu\sigma}\mathcal{F}_\nu{}^\sigma$$

where Ric is Ricci tensor, R the scalar curvature and $\mathcal{F} = \mathrm{d}A$ the electromagnetic tensor.

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Under some assumptions on M, Q and Λ, F has four roots $r_n < 0 < r_c < r_- < r_+$ and is positive between r_- and r_+. The **exterior DSRN spacetime** is the lorentzian manifold $\mathcal{M} = \mathbb{R}_t \times]r_-, r_+[_r \times \mathbb{S}^2_\omega$ endowed with the metric g.

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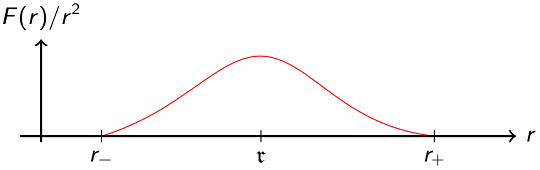
$\rightsquigarrow$ **Photon sphere** $\mathbb{R}_t \times \{\mathfrak{r}\}_r \times \mathbb{S}^2_\omega$.



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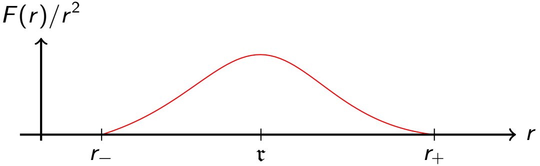

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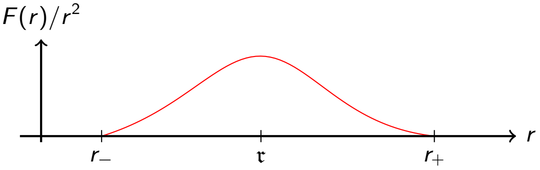

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Charged Klein-Gordon (KG) equation, quadratic pencil

Let $q \in \mathbb{R}$ and $m > 0$. We define the charged wave operator on $(\mathcal{M}, g)$

$$\Box_g := (\nabla_\mu - \mathrm{i}qA_\mu)(\nabla^\mu - \mathrm{i}qA^\mu)$$

and the charged KG equation reads

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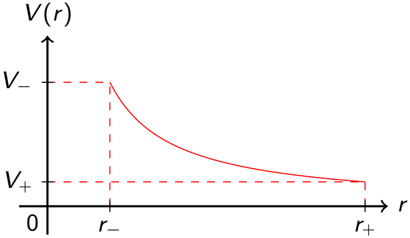

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If $u(t,x,\omega) = \mathrm{e}^{\mathrm{i}zt}v(x,\omega)$ solves $(\partial_t - \mathrm{i}sV)^2 u + Pu = 0$ for some $z \in \mathbb{C}$, then $Pv - (z - sV)^2 v = 0$. This motivates us to define the **quadratic pencil**

$$p(z,s) := P - (z - sV)^2.$$

Superradiance

For $\ell \in \mathbb{R}$, the energies

$$\langle u \,|\, u \rangle_\ell := \|(\partial_t - \mathrm{i}\ell)u\|^2 + \langle (P - (sV - \ell)^2)u, u \rangle$$

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- Reissner-Nordström ($\Lambda = 0$): $F(r) \to 1$ as $r \to +\infty$, so for s **small**

$$\langle u \,|\, u \rangle_{sV_-} \gtrsim \langle -r^{-1}\partial_x r^2 \partial_x r^{-1} u, u\rangle + \langle -r^{-2} F \Delta_{\mathbb{S}^2} u, u\rangle$$
$$+ \langle (m^2 F - s^2 (V - V_-)^2) u, u\rangle > 0$$

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- DSRN ($\Lambda > 0$): **these energies are never positive**.

  Fix ℓ so that $sV_- - \ell \neq 0$, let f supported in $]-\infty, -1[$ such that $\|(\partial_t - \mathrm{i}\ell) f\| = 0$ and $-\Delta_{\mathbb{S}^2} f = 0$, and choose $R \gg 0$:

$$\langle R^{-1/2} f(x/R) \,|\, R^{-1/2} f(x/R) \rangle_\ell$$
$$= R^{-2}\|R^{-1} f'(x/R)\|^2 + \langle (r^{-1}FF' + m^2 F - (sV - \ell)^2) R^{-1/2} f(x/R), R^{-1/2} f(x/R)\rangle$$
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- Kerr's type topology (following the rotation):

$$\|(u_0, u_1)\|^2_{\dot{\mathcal{E}}} := \langle Pu_0, u_0\rangle + \|u_1 - sVu_0\|^2.$$

Charge Klein-Gordon operator

If u solves the second order equation

$$(\partial_t - \mathrm{i}sV)^2 u + Pu = 0, \tag{2}$$

then $v := (u, -\mathrm{i}\partial_t u - sVu)$ solves the first order equation

$$-\mathrm{i}\partial_t v = \hat{K}(s)v, \qquad \hat{K}(s) = \begin{pmatrix} sV & \mathrm{Id} \\ P & sV \end{pmatrix}. \tag{3}$$

Conversely, if $v = (v_0, v_1)$ solves (3), then v_0 solves (2). $\hat{K}(s)$ is the **charge Klein-Gordon operator**.

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- Link between charge KG operator and quadratic pencil: $\forall z \in \rho(\hat{K}(s))$

$$(\hat{K}(s) - z)^{-1} = \begin{pmatrix} p(z,s)^{-1}(z - sV) & p(z,s)^{-1} \\ \mathrm{Id} + (z - sV)\, p(z,s)^{-1}(z - sV) & (z - sV)\, p(z,s)^{-1} \end{pmatrix}.$$

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- Scattering theory without positive conserved energy (Kako, Bachelot, Gérard), Georgescu-Gérard-Häfner (De Sitter-Kerr), Dafermos-Rodnianski-Shlapentokh-Rothman (Kerr).

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$$\hat{H}(s) := \begin{pmatrix} 0 & \mathrm{Id} \\ P - s^2 V^2 & 2sV \end{pmatrix}, \qquad \hat{H}_\pm(s) := \begin{pmatrix} 0 & \mathrm{Id} \\ P - s^2 \tilde{V}_\pm^2 & 2s\tilde{V}_\pm \end{pmatrix}.$$

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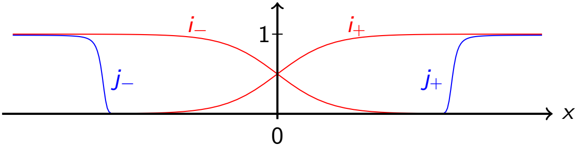

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- $\hat{H}_\pm(s)$ defined on the following energy spaces ($\mathcal{H} = L^2(\mathbb{R}\times\mathbb{S}^2, \mathrm{d}x\mathrm{d}\omega)$)

$$\dot{\mathcal{E}}_+ := (P - s^2\tilde{V}_+^2)^{-1/2}\mathcal{H} \oplus \mathcal{H},$$
$$\dot{\mathcal{E}}_- := \Phi(sV_-)^{-1}\left((P - s^2(\tilde{V}_- - V_-))^{-1/2}\mathcal{H} \oplus \mathcal{H}\right).$$

Proposition (Georgescu-Gérard-Häfner, 2017)

There exists $\varepsilon > 0$ such that, for all $\delta > 0$, $w^\delta(\hat{H}(s) - z)^{-1}w^\delta$ and $w^\delta(\hat{H}_\pm(s) - z)^{-1}w^\delta$ extend meromorphically from $\mathbb{C}^+$ to $\{\omega \in \mathbb{C} \mid \Im\omega \geq \varepsilon\}$ as compact operators. The poles are called **resonances** (noted $\mathrm{Res}(p)$).

Meromorphic extension

- We introduce the operator
$$Q(s,z) := \sum_{\pm} i_{\pm}^2 (\hat{H}_{\pm}(s) - z)^{-1}.$$
Then
$$w^{\delta} Q(s,z) w^{\delta} = w^{\delta} (\hat{H}(s) - z)^{-1} w^{\delta} (\mathrm{Id} + \mathcal{K}(s,z))$$
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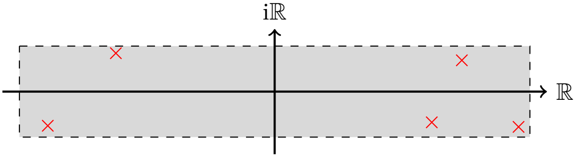
- By multidimensional analytic Fredholm theory, there exist $\varepsilon > 0$ and a subvariety $S \subset D(0,1) \times \{|\Re z| < R, |\Im z| < \varepsilon\}$ such that $\mathrm{Id} + \mathcal{K}(s,z)$ is invertible on $D(0,1) \times \{|\Re z| < R, |\Im z| < \varepsilon\} \setminus S$.



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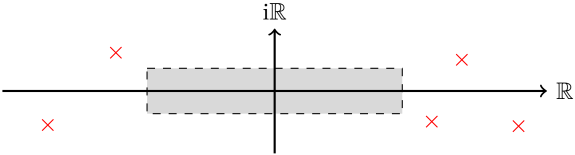

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$\Longrightarrow$ Resonances of $\hat{H}(s)$ are those of $\hat{H}_{\pm}(s)$ near 0.

Localization of high frequency resonances

Extension of Sà-Barreto-Zworski, Sjöstrand results:

Proposition (B., 2018)

There exist $K > 0$ and $\theta > 0$ such that, for any $C > 0$, there exists an injective map $\tilde{b} : \Gamma \to \mathrm{Res}(p)$ with

$$\Gamma = \frac{\sqrt{F(\mathfrak{r})}}{\mathfrak{r}} \left(\pm\mathbb{N} \setminus \{0\} \pm \frac{1}{2} + \frac{qQ}{\sqrt{F(\mathfrak{r})}} - \frac{\mathrm{i}}{2} \sqrt{\left| 3 - \frac{12M}{\mathfrak{r}} + \frac{10Q^2}{\mathfrak{r}^2} \right|} \left(\mathbb{N} + \frac{1}{2} \right) \right)$$

the set of pseudo-poles, such that all the poles in

$$\Omega_C = \{ \lambda \in \mathbb{C} \mid |\lambda| > K, \Im\lambda > -\max\{C, \theta|\Re\lambda|\} \}$$

are in the image of $\tilde{b}$. Furthermore, if $\mu \in \Gamma$ and $\tilde{b}(\mu) \in \Omega_C$, then

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- $\hat{R}(z,s) := (\hat{K}(s) - z)^{-1}$. $\forall z_0 \in \mathrm{Res}(p)$ of multiplicity $m(z_0)$, $\forall k > -m(z_0)$

$$\Pi^{\chi}_{j,k} := \frac{1}{2\pi\mathrm{i}} \oint \frac{(-\mathrm{i})^k}{k!} \chi \hat{R}(z) \chi (z - z_0)^k \mathrm{d}z \qquad \chi \in \mathcal{C}^\infty_c(\mathbb{R}, \mathbb{R}).$$

Resonances expansion of the local propagator

Theorem (B., 2018)

1. There exist $\delta, N > 0$ and a discrete set $\mathscr{S} \subset \mathbb{C}$ such that for all $\nu \in \mathbb{R} \setminus \mathscr{S}$ with $0 < \nu < \delta$ and for all $u \in \dot{\mathcal{E}}$ with $\langle -\Delta_{\mathbb{S}^2} \rangle^N u \in \dot{\mathcal{E}}$, we have for s **small** enough and $t \gg 0$

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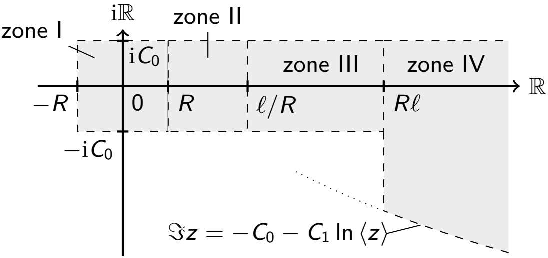
- Extension of Bony-Häfner result for the wave equation in De Sitter-Schwarzschild spacetime : adaptation of the arguments by perturbation for s **small**.
- Part 2. $\Longrightarrow$ **integrability** of local energy if $(\ln\langle -\Delta_{\mathbb{S}^2} \rangle)^{\alpha} u \in \dot{\mathcal{E}}$ for some $\alpha > 1$, **exponential decay** if $\langle -\Delta_{\mathbb{S}^2} \rangle^{\beta} u \in \dot{\mathcal{E}}$ for some $\beta > 0$.

Frequency regions

$-\Delta_{\mathbb{S}^2} \rightsquigarrow \ell(\ell+1)$. The proof is based on estimates for $\chi p_\ell(z,s)^{-1}\chi$ uniformly in $\ell \in \mathbb{N}$. We distinguish four regimes:



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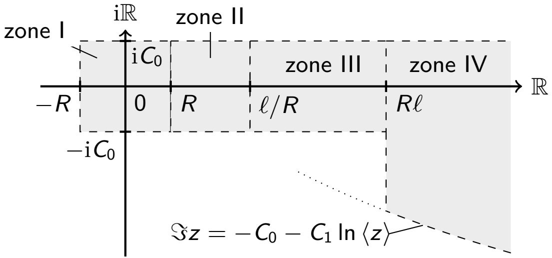

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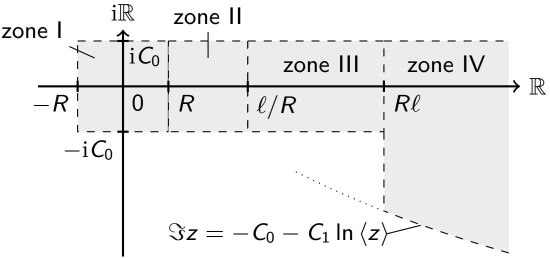

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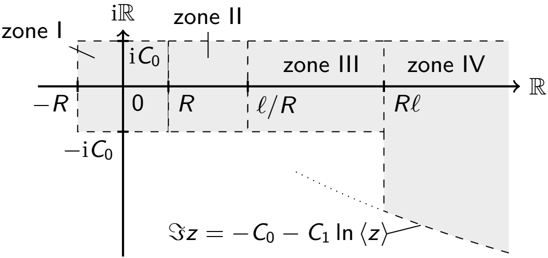

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- Zone IV (high frequencies): semiclassical limiting absorption principle for quadratic pencil + estimate with F.B.I. transform (Martinez).

Contour deformation

Let $\mu, \nu > 0$. We have

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as operators from $\dot{\mathcal{E}}$ to $\dot{\mathcal{E}}^{-2} := (\hat{K}(s) - \omega)^2 \dot{\mathcal{E}}$ ($\Im\omega$ sufficiently large). Then

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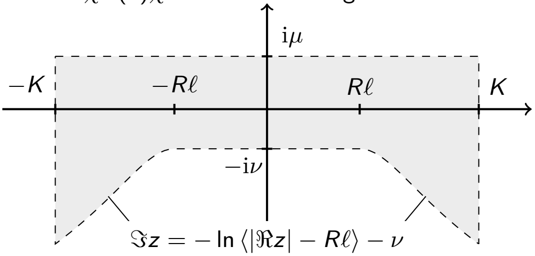
1. We integrate $e^{-izt}\chi\hat{R}(z)\chi$ over the following contour:



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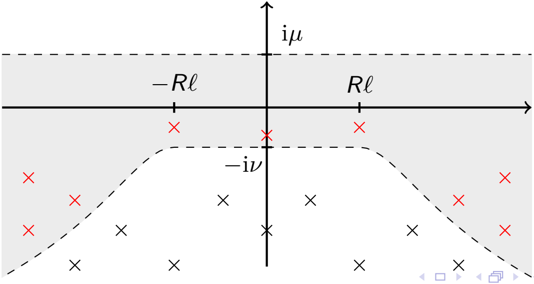
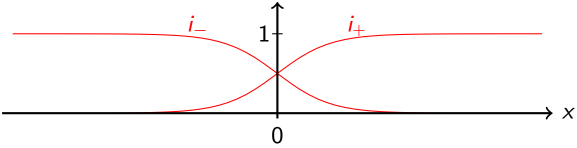
2. Letting $K \to +\infty$ and using the above estimates on $\chi\hat{R}_\ell(z)\chi$ and $\chi p_\ell(z,s)^{-1}\chi$ as well as residue theorem gives the expansion.



Asymptotic dynamics

Let $i_\pm \in \mathcal{C}^\infty(\mathbb{R}, [0,1])$ as below:



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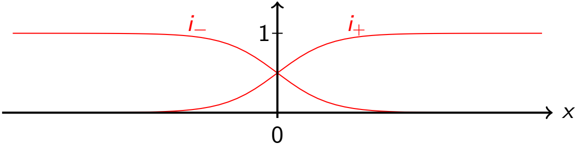

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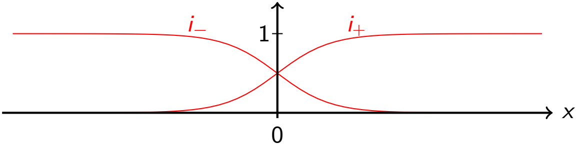

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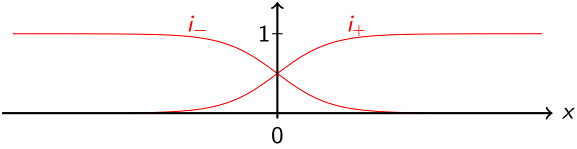

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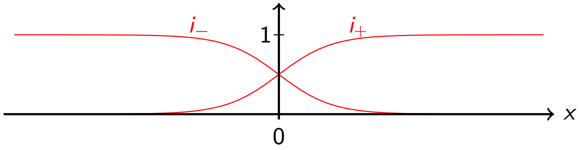
$$\hat{H}_\pm(s) := \begin{pmatrix} 0 & \mathrm{Id} \\ P - s^2 V_\pm^2 & 2sV_\pm \end{pmatrix} \quad \text{on} \quad \dot{\mathcal{E}}_\pm := \overline{\mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{S}^2) \times \mathcal{C}_c^\infty(\mathbb{R}\times\mathbb{S}^2)}^{\langle\cdot|\cdot\rangle_{sV_\pm}}$$

Asymptotic dynamics

Let $i_\pm \in \mathcal{C}^\infty(\mathbb{R}, [0,1])$ as below:



- Asymptotic equations:

$$(\partial_t - \mathrm{i}sV_\pm)^2 u - \partial_x^2 u = 0. \tag{4}$$

- **Positive** conserved energies: for u solution of (4),

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- For all $\ell \in \mathbb{N}$, set $Z_\ell := \mathbb{1}_{\ell(\ell+1)}(-\Delta_{\mathbb{S}^2})L^2(\mathbb{R}\times\mathbb{S}^2, \mathrm{d}x\mathrm{d}\omega)$ and let

$$W_\ell := (Z_\ell \otimes L^2(\mathbb{R}, \mathrm{d}x))^{\oplus 2}, \qquad \dot{\mathcal{E}}_\pm^{\mathrm{fin}} := \Big\{ u \in \dot{\mathcal{E}}_\pm \mid \exists L > 0,\ u \in \bigoplus_{\ell \le L} (\dot{\mathcal{E}}_\pm \cap W_\ell) \Big\}.$$

Asymptotic completeness

Theorem (Georgescu-Gérard-Häfner, 2017)

Assume s sufficiently small.

1. For all $u \in \dot{\mathcal{E}}_{\pm}^{\mathrm{fin}}$, the limits
$$W_{\pm}(s)u = \lim_{t \to +\infty} \mathrm{e}^{\mathrm{i}t\hat{H}(s)} i_{\pm}^{2} \mathrm{e}^{-\mathrm{i}t\hat{H}_{\pm}(s)} u$$
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- Existence and completeness of wave operators for the wave equation in Kerr spacetime with no angular restriction by Dafermos-Rodnianski-Shlapentokh-Rothman.

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- Numerical investigation of localization of low frequency resonances in De Sitter-Reissner-Nordström spacetime in progress.

Thank you for your attention!