Decay of the local energy and scattering theory for the charged Klein-Gordon equation in the exterior De Sitter-Reissner-Nordström spacetime

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The exterior De Sitter-Reissner-Nordström (DSRN) spacetime

In Boyer-Lindquist coordinates \((t, r, \omega) \in \mathbb{R}_t \times ]0, +\infty[ \times S^2_\omega\), the DSRN metric \(g\) is given by

\[
g = F(r)dt^2 - F(r)^{-1}dr^2 - r^2d\omega^2, \quad F(r) = 1 - \frac{2M}{r} + \frac{Q^2}{2r^2} - \frac{\Lambda r^2}{3}
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where \(M > 0\) is the black hole mass, \(Q \neq 0\) its electric charge and \(\Lambda > 0\) is the cosmological constant.
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Let \( A_\mu = \frac{Q}{r} dt \). Then \( (g, A_\mu) \) solves the **Einstein-Maxwell field equation**

\[
\text{Ric}_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu}, \quad T_{\mu\nu} = \frac{1}{4} g_{\mu\nu} F^{\sigma\rho} F_{\sigma\rho} - F_{\mu\sigma} F_{\nu}^{\sigma}
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where \( \text{Ric} \) is Ricci tensor, \( R \) the scalar curvature and \( F = dA \) the electromagnetic tensor.
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Under some assumptions on \(M, Q\) and \(\Lambda\), \(F\) has four roots \(r_n < 0 < r_c < r_- < r_+\) and is positive between \(r_-\) and \(r_+\). The exterior DSRN spacetime is the lorentzian manifold \(\mathcal{M} = \mathbb{R}_t \times ]r_-, r_+[, \, r_+[ \times S^2_\omega\) endowed with the metric \(g\).
Geometric properties of the exterior DSRN spacetime

- Eternal, spherical, static and charged black hole.

\[ \begin{align*}
\frac{\partial}{\partial t} (\text{timelike}) + & \text{2 others (spherical symmetry).} \\
\text{Noether's theorem} + \text{mass equation} \Rightarrow & 4 \text{ conserved quantities for the geodesic motion: for null geodesic,} \\
g(\dot{\gamma}, \partial_t) = & E \in \mathbb{R}, \\
g(\dot{\gamma}, \partial_\phi) = & L \in \mathbb{R}, \\
g(\dot{\gamma}, \dot{\gamma}) = & 0. \\
\end{align*} \]

\[ E^2 = \dot{r}^2 + F(r) r^2 \frac{L^2}{2}, \]

Trivial solution for \( r = r^\pm \) at which \( F(r)/r^2 \) is maximal \( \Rightarrow \) Photon sphere \( \mathbb{R}_t \times \{ r \} \times S^2 \).

This is the only photon sphere in the exterior DSRN spacetime (4 roots for \( F \), 1 photon sphere: see Mokdad).
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- 3 Killing vector fields: $\partial_t$ (timelike) + 2 others (spherical symmetry).
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  $\sim$ Photon sphere $\mathbb{R}_t \times \{t\}_r \times S^2_\omega.$

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Charged Klein-Gordon (KG) equation, quadratic pencil

Let $q \in \mathbb{R}$ and $m > 0$. We define the charged wave operator on $(\mathcal{M}, g)$

$$\Box_g := (\nabla_\mu - iqA_\mu)(\nabla^\mu - iqA^\mu)$$

and the charged KG equation reads

$$\Box_g u + m^2 u = 0.$$  \hfill (1)
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Set $s := qQ \in \mathbb{R}$ the charge product and $V(r) := r^{-1}$ so that (1) becomes

$$(\partial_t - isV)^2 u + \hat{P} u = 0, \quad \hat{P} = -\frac{F}{r^2} \partial_r (r^2 F \partial_r) - \frac{F}{r^2} \Delta_{S^2} + m^2 F.$$
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We introduce the operator $P := r \hat{P} r^{-1}$ and the **Regge-Wheeler coordinate**

$$\frac{dx}{dr} := \frac{1}{F(r)}, \quad x(t) = 0 \quad \text{(maximum of } F/r^2).$$
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Principal symbol of \( P \): \( \xi^2 + \eta^2 W(x), \ W = F/r^2 \rightsquigarrow \text{normally hyperbolic trapping.} \)
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If \( u(t, x, \omega) = e^{izt} v(x, \omega) \) solves \((\partial_t - isV)^2 u + Pu = 0\) for some \( z \in \mathbb{C} \), then \( P\nu - (z - sV)^2 \nu = 0 \). This motivates us to define the quadratic pencil
\[
p(z, s) := P - (z - sV)^2.
\]
Superradiance

For $\ell \in \mathbb{R}$, the energies

$$\langle u | u \rangle_{\ell} := \| (\partial_t - i \ell) u \|^2 + \langle (P - (sV - \ell)^2) u, u \rangle$$

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- **Reissner-Nordström ($\Lambda = 0$):** $F(r) \to 1$ as $r \to +\infty$, so for $s$ small
  \[
  \langle u \mid u \rangle_{sV_-} \gtrsim \langle -r^{-1} \partial_x r^2 \partial_x r^{-1} u, u \rangle + \langle -r^{-2} F \Delta_{S^2} u, u \rangle
  + \langle (m^2 F - s^2 (V - V_-)^2) u, u \rangle > 0
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  because $F(r), V(r) - V_- = O_{r \to r_-} (r - r_-)$.
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- **DSRN ($\Lambda > 0$):** these energies are never positive.

  Fix $\ell$ so that $sV_- - \ell \neq 0$, let $f$ supported in $]-\infty, -1[\,$ such that

  $$\| (\partial_t - i\ell)f \| = 0$$

  and $-\Delta_{S^2} f = 0$, and choose $R \gg 0$:

  $$\langle R^{-1/2}f(x/R) \mid R^{-1/2}f(x/R) \rangle_\ell$$

  $$= R^{-2}\| R^{-1}f'(x/R) \|^2 + \langle (r^{-1}FF' + m^2 F - (sV - \ell)^2)R^{-1/2}f(x/R), R^{-1/2}f(x/R) \rangle$$

  $$= R^{-2}\| f' \|^2 - (sV_- - \ell)^2 \left(1 + o_{R \to +\infty}(1)\right)\| f \|^2 < 0.$$
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  \[ \langle R^{-1/2}f(x/R) | R^{-1/2}f(x/R) \rangle_\ell \]
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- Kerr's type topology (following the rotation):
  \[ \| (u_0, u_1) \|_{\tilde{g}}^2 := \langle Pu_0, u_0 \rangle + \| u_1 - sVu_0 \|^2. \]
Charge Klein-Gordon operator

If $u$ solves the second order equation

\[(\partial_t - isV)^2 u + Pu = 0,\]  \hspace{1cm} (2)

then $v := (u, -i\partial_t u - sVu)$ solves the first order equation

\[-i\partial_t v = \hat{K}(s)v, \hspace{1cm} \hat{K}(s) = \begin{pmatrix} sV & Id \\ P & sV \end{pmatrix}.\]  \hspace{1cm} (3)

Conversely, if $v = (v_0, v_1)$ solves (3), then $v_0$ solves (2). $\hat{K}(s)$ is the charge Klein-Gordon operator.
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- **Homogeneous energy space** \( \mathcal{E} := \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{S}^2) \times \mathcal{C}_c^\infty(\mathbb{R} \times \mathbb{S}^2) \| \cdot \|_{\mathcal{E}} \).
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- \( \hat{K}(s) \) generates a continuous semi-group \( (e^{it\hat{K}(s)})_{t \in \mathbb{R}} \) on \( (\dot{\mathcal{E}}, \| \cdot \|_{\dot{\mathcal{E}}}) \).
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- $\hat{K}(s)$ generates a continuous semi-group $(e^{it\hat{K}(s)})_{t \in \mathbb{R}}$ on $(\mathcal{E}, \|\cdot\|_\mathcal{E})$.  
- Link between charge KG operator and quadratic pencil: $\forall z \in \rho(\hat{K}(s))$
\[
(\hat{K}(s) - z)^{-1} = \begin{pmatrix} p(z, s)^{-1}(z - sV) & p(z, s)^{-1} \\ \text{Id} + (z - sV)p(z, s)^{-1}(z - sV) & (z - sV)p(z, s)^{-1} \end{pmatrix}.
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Localization of resonances:
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Meromorphic extension

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\[ u \mapsto e^{-isV_+^t} u \mapsto \lim_{r \to r_+} V(r) = 0. \]

Let \( w(x) := \sqrt{(r - r_-)(r_+ - r)} \), \( i_\pm, j_\pm \in C^\infty(\mathbb{R}, [0, 1]) \) as below, \( \tilde{V}_\pm := V \mp j_\pm^2 r_-^{-1} \) and

\[
\hat{H}(s) := \begin{pmatrix}
0 & \text{Id} \\
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\end{pmatrix}, \quad \hat{H}_\pm(s) := \begin{pmatrix}
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\end{pmatrix}.
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\( \hat{H}(s) = \Phi(sV)\hat{K}(s)\Phi(sV)^{-1}, \Phi(sV) \) isomorphism on \( \mathcal{E} \).

\[ x \quad \begin{array}{c}
i_-\\
\mid \\
0 \mid \\
\mid \\
1 \mid \\
\mid \\
i_+\\
\end{array} \quad \begin{array}{c}
j_-\\n\mid \\
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\mid \\
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\]

\( \hat{H}(s) = \Phi(sV)\hat{K}(s)\Phi(sV)^{-1}, \Phi(sV) \) isomorphism on \( \hat{E} \).

\( \hat{H}_\pm(s) \) defined on the following energy spaces (\( \mathcal{H} = L^2(\mathbb{R} \times S^2, dx d\omega) \))

\[
\hat{E}_+ := (P - s^2 \tilde{V}^2_+)^{-1/2} \mathcal{H} \oplus \mathcal{H},
\]

\[
\hat{E}_- := \Phi(sV_-)^{-1} ((P - s^2(\tilde{V}_- - V_-))^{-1/2} \mathcal{H} \oplus \mathcal{H}).
\]

**Proposition (Georgescu-Gérard-Häfner, 2017)**

There exists \( \varepsilon > 0 \) such that, for all \( \delta > 0 \), \( w^\delta(\hat{H}(s) - z)^{-1}w^\delta \) and \( w^\delta(\hat{H}_\pm(s) - z)^{-1}w^\delta \) extend meromorphically from \( \mathbb{C}^+ \) to \( \{ \omega \in \mathbb{C} \mid \Im \omega \geq \varepsilon \} \) as compact operators. The poles are called **resonances** (noted \( \text{Res}(p) \)).
We introduce the operator

\[ Q(s, z) := \sum_{\pm} i_{\pm}^2 (\hat{H}_{\pm}(s) - z)^{-1}. \]

Then

\[ w^\delta Q(s, z)w^\delta = w^\delta (\hat{H}(s) - z)^{-1}w^\delta (\text{Id} + \mathcal{K}(s, z)) \]

where \( \mathcal{K}(s, z) \) is compact. Since \( \hat{H}(0) = \hat{H}_\pm(0) \), we have \( \mathcal{K}(0, z) = 0 \).
We introduce the operator
\[ Q(s, z) := \sum_{\pm} i^{2\pm}(\hat{H}_{\pm}(s) - z)^{-1}. \]
Then
\[ w^\delta Q(s, z)w^\delta = w^\delta (\hat{H}(s) - z)^{-1}w^\delta (\text{Id} + \mathcal{K}(s, z)) \]
where \( \mathcal{K}(s, z) \) is compact. Since \( \hat{H}(0) = \hat{H}_{\pm}(0) \), we have \( \mathcal{K}(0, z) = 0 \).

By multidimensional analytic Fredholm theory, there exist \( \varepsilon > 0 \) and a subvariety \( S \subset D(0, 1) \times \{ |\Re z| < R, |\Im z| < \varepsilon \} \) such that \( \text{Id} + \mathcal{K}(s, z) \) is invertible on \( D(0, 1) \times \{ |\Re z| < R, |\Im z| < \varepsilon \} \setminus S \).
Meromorphic extension

We introduce the operator

\[ Q(s, z) := \sum_{\pm} i^{2} (\hat{H}_{\pm}(s) - z)^{-1}. \]

Then

\[ w^{\delta} Q(s, z) w^{\delta} = w^{\delta} (\hat{H}(s) - z)^{-1} w^{\delta} (\text{Id} + \mathcal{K}(s, z)) \]

where \( \mathcal{K}(s, z) \) is compact. Since \( \hat{H}(0) = \hat{H}_{\pm}(0) \), we have \( \mathcal{K}(0, z) = 0 \).

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Assume that \( (s_n, z_n) \in D(0, 1/n) \times \{ |\Re z| < R/2, |\Im z| < \varepsilon/2 \} \) is such that \( (s_n, z_n) \to (0, z_0) \) and \( \text{Id} + \mathcal{K}(s_n, z_n) \) is not invertible for all \( n \geq 1 \). But \( \text{Id} + \mathcal{K}(0, z_0) = \text{Id} \) is invertible, a contradiction.
Meromorphic extension

We introduce the operator
\[ Q(s, z) := \sum_{\pm} i^2(\hat{H}_{\pm}(s) - z)^{-1}. \]

Then
\[ w^\delta Q(s, z)w^\delta = w^\delta (\hat{H}(s) - z)^{-1}w^\delta (\text{Id} + K(s, z)) \]
where \( K(s, z) \) is compact. Since \( \hat{H}(0) = \hat{H}_{\pm}(0) \), we have \( K(0, z) = 0 \).

By multidimensional analytic Fredholm theory, there exist \( \varepsilon > 0 \) and a subvariety \( S \subset D(0, 1) \times \{|\Re z| < R, |\Im z| < \varepsilon\} \) such that \( \text{Id} + K(s, z) \) is invertible on \( D(0, 1) \times \{|\Re z| < R, |\Im z| < \varepsilon\} \setminus S \).

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For \( (s, z) \in D(0, s_0) \times \{|\Re z| < R/2, |\Im z| < \varepsilon/2\} \) with \( s_0 \) small enough,
\[ w^\delta Q(s, z)w^\delta (\text{Id} + K(s, z))^{-1} = w^\delta (\hat{H}(s) - z)^{-1}w^\delta \]

Resonances of \( \hat{H}(s) \) are those of \( \hat{H}_{\pm}(s) \) near 0.
Meromorphic extension

We introduce the operator

\[ Q(s, z) := \sum_{\pm} i^2_{\pm} (\hat{H}_{\pm}(s) - z)^{-1}. \]

Then

\[ w^\delta Q(s, z)w^\delta = w^\delta (\hat{H}(s) - z)^{-1}w^\delta (\text{Id} + \mathcal{K}(s, z)) \]

where \( \mathcal{K}(s, z) \) is compact. Since \( \hat{H}(0) = \hat{H}_{\pm}(0) \), we have \( \mathcal{K}(0, z) = 0 \).

By multidimensional analytic Fredholm theory, there exist \( \varepsilon > 0 \) and a subvariety \( S \subset D(0, 1) \times \{|\Re z| < R, |\Im z| < \varepsilon\} \) such that \( \text{Id} + \mathcal{K}(s, z) \) is invertible on \( D(0, 1) \times \{|\Re z| < R, |\Im z| < \varepsilon\} \setminus S \).

Assume that \( (s_n, z_n) \in D(0, 1/n) \times \{|\Re z| < R/2, |\Im z| < \varepsilon/2\} \) is such that \( (s_n, z_n) \to (0, z_0) \) and \( \text{Id} + \mathcal{K}(s_n, z_n) \) is not invertible for all \( n \geq 1 \). But \( \text{Id} + \mathcal{K}(0, z_0) = \text{Id} \) is invertible, a contradiction.

For \( (s, z) \in D(0, s_0) \times \{|\Re z| < R/2, |\Im z| < \varepsilon/2\} \) with \( s_0 \) small enough,

\[ w^\delta Q(s, z)w^\delta (\text{Id} + \mathcal{K}(s, z))^{-1} = w^\delta (\hat{H}(s) - z)^{-1}w^\delta \]

\[ \implies \text{Resonances of } \hat{H}(s) \text{ are those of } \hat{H}_{\pm}(s) \text{ near } 0. \]
Localization of high frequency resonances

Extension of Sà-Barreto-Zworski, Sjöstrand results:

Proposition (B., 2018)

There exist $K > 0$ and $\theta > 0$ such that, for any $C > 0$, there exists an injective map $\tilde{b} : \Gamma \to \text{Res}(p)$ with

$$\Gamma = \frac{\sqrt{F(\tau)}}{r} \left( \pm N \setminus \{0\} \pm \frac{1}{2} + \frac{qQ}{\sqrt{F(\tau)}} - \frac{i}{2} \sqrt{3 - \frac{12M}{r} + \frac{10Q^2}{r^2}} \right)^{\left( N + \frac{1}{2} \right)}$$

the set of pseudo-poles, such that all the poles in

$$\Omega_C = \{ \lambda \in \mathbb{C} \mid |\lambda| > K, \Im \lambda > -\max\{C, \theta|\Re \lambda|\} \}$$

are in the image of $\tilde{b}$. Furthermore, if $\mu \in \Gamma$ and $\tilde{b}(\mu) \in \Omega_C$, then

$$\lim_{|\mu| \to +\infty} (\tilde{b}(\mu) - \mu) = 0.$$
Localization of high frequency resonances

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**Semicalssical problem:** $p_h(\sqrt{z}, s) = -\hbar^2 \partial_x^2 + W(h) - (\sqrt{z} - h s V)^2 \implies \text{shift:}$ $z$ pseudo-pole for wave operator, $(\sqrt{z} - h s V(0))^2$ pseudo-pole for $P_h$. 
Localization of high frequency resonances

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the set of pseudo-poles, such that all the poles in

$$\Omega_C = \{ \lambda \in \mathbb{C} \mid |\lambda| > K, \Im \lambda > -\max\{C, \theta |\Re \lambda|\} \}$$

are in the image of $\tilde{b}$. Furthermore, if $\mu \in \Gamma$ and $\tilde{b}(\mu) \in \Omega_C$, then

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**Semicalssical problem:** $p_h(\sqrt{z}, s) = -h^2 \partial_x^2 + W(h) - (\sqrt{z} - hsV)^2 \implies$ **shift:** $z$ pseudo-pole for wave operator, $(\sqrt{z} - hsV(0))^2$ pseudo-pole for $P_h$.

**$\hat{R}(z, s) := (\hat{K}(s) - z)^{-1}$.** $\forall z_0 \in \text{Res}(p)$ of multiplicity $m(z_0)$, $\forall k > -m(z_0)$

$$\Pi^\chi_{j,k} := \frac{1}{2\pi i} \oint (-i)^k \frac{1}{k!} \chi \hat{R}(z) \chi(z - z_0)^k \, dz$$

$\chi \in C_c^\infty(\mathbb{R}, \mathbb{R}).$
Resonances expansion of the local propagator

**Theorem (B., 2018)**

1. There exist $\delta, N > 0$ and a discrete set $\mathcal{I} \subset \mathbb{C}$ such that for all $\nu \in \mathbb{R} \setminus \mathcal{I}$ with $0 < \nu < \delta$ and for all $u \in \mathcal{E}$ with $\langle -\Delta_{S^2} \rangle^N u \in \mathcal{E}$, we have for $s$ small enough and $t \gg 0$

$$\chi e^{-it\hat{K}} \chi u = \sum_{z_j \in \text{Res}(p)} \sum_{k=0}^{m(z_j)} e^{-iz_j t} t^k \nabla_{j,k} u + \mathcal{O} \left( e^{-\nu t} \| \langle -\Delta_{S^2} \rangle^N u \|_{\mathcal{E}} \right).$$

2. There exists $\varepsilon > 0$ such that, for any increasing positive function $g$ with $\lim_{x \to +\infty} g(x) = +\infty$ and $g(x) \leq x$ for $x \gg 0$, for all $u \in \mathcal{E}$ with $g(-\Delta_{S^2}) u \in \mathcal{E}$ and $s$ small enough, we have for $t \gg 0$

$$\| \chi e^{-it\hat{K}} \chi u \|_{\mathcal{E}} \lesssim \left( g\left( e^{\varepsilon t} \right) \right)^{-1} \| g(-\Delta_{S^2}) u \|_{\mathcal{E}}.$$
Resonances expansion of the local propagator

Theorem (B., 2018)

1. There exist $\delta, N > 0$ and a discrete set $\mathcal{I} \subset \mathbb{C}$ such that for all $\nu \in \mathbb{R} \setminus \mathcal{I}$ with $0 < \nu < \delta$ and for all $u \in \mathcal{E}$ with $\langle -\Delta_{S^2} \rangle^N u \in \mathcal{E}$, we have for $s$ small enough and $t \gg 0$

$$\chi e^{-it\hat{K}} \chi u = \sum_{z_j \in \text{Res}(p)} \sum_{k=0}^{m(z_j)} e^{-iz_j t} t^k \prod_{j,k}^\chi u + \mathcal{O} \left( e^{-\nu t} \| \langle -\Delta_{S^2} \rangle^N u \|_{\dot{\mathcal{E}}} \right).$$

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$$\| \chi e^{-it\hat{K}} \chi u \|_{\dot{\mathcal{E}}} \leq (g(e^{\varepsilon t}))^{-1} \| g(-\Delta_{S^2}) u \|_{\dot{\mathcal{E}}}. $$

Extension of Bony-Hfäner result for the wave equation in De Sitter-Schwarzschild spacetime : adaptation of the arguments by perturbation for $s$ small.
Resonances expansion of the local propagator

Theorem (B., 2018)

1. There exist $\delta, N > 0$ and a discrete set $\mathcal{I} \subset \mathbb{C}$ such that for all $\nu \in \mathbb{R} \setminus \mathcal{I}$ with $0 < \nu < \delta$ and for all $u \in \mathcal{E}$ with $\langle -\Delta_{S^2} \rangle^N u \in \mathcal{E}$, we have for $s$ small enough and $t \gg 0$

$$\chi e^{-it\hat{K}} \chi u = \sum_{z_j \in \text{Res}(p)} \sum_{k=0}^{m(z_j)} e^{-iz_j t} t^k \prod_{j,k} \chi u + O \left( e^{-\nu t} \| \langle -\Delta_{S^2} \rangle^N u \|_{\mathcal{E}} \right).$$

2. There exists $\varepsilon > 0$ such that, for any increasing positive function $g$ with $\lim_{x \to +\infty} g(x) = +\infty$ and $g(x) \leq x$ for $x \gg 0$, for all $u \in \mathcal{E}$ with $g(-\Delta_{S^2}) u \in \mathcal{E}$ and $s$ small enough, we have for $t \gg 0$

$$\| \chi e^{-it\hat{K}} \chi u \|_{\mathcal{E}} \lesssim (g(e^{\varepsilon t}))^{-1} g(-\Delta_{S^2}) u \|_{\mathcal{E}}.$$
Frequency regions

\(-\Delta_{S^2} \sim \ell (\ell + 1)\). The proof is based on estimates for \(\chi p_\ell (z, s)^{-1} \chi\) uniformly in \(\ell \in \mathbb{N}\). We distinguish four regimes:

- Zone I (low frequencies): multidimensional Fredholm theory + Georgescu-Gérard-Häfner + study of the frequency 0 (Bony-Häfner, Bachelot).
- Zone II (lifting zone): complex scaling (Zworski).
- Zone IV (high frequencies): semiclassical limiting absorption principle for quadratic pencil + estimate with F.B.I. transform (Martinez).

\[ \Im z = -C_0 - C_1 \ln \langle z \rangle \]
Frequency regions

\[-\Delta S_2 \rightsquigarrow \ell (\ell + 1).\] The proof is based on estimates for \(\chi p_\ell (z, s)^{-1} \chi\) uniformly in \(\ell \in \mathbb{N}\). We distinguish four regimes:

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Frequency regions

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Frequency regions

$-\Delta S^2 \sim \ell (\ell + 1)$. The proof is based on estimates for $\chi p_\ell(z, s)^{-1} \chi$ uniformly in $\ell \in \mathbb{N}$. We distinguish four regimes:

- **Zone I (low frequencies):** multidimensional Fredholm theory + Georgescu-Gérard-Häfner + study of the frequency 0 (Bony-Häfner, Bachelot).
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\[ \Im z = -C_0 - C_1 \ln \langle z \rangle \]
Contour deformation

Let $\mu, \nu > 0$. We have

$$\chi e^{-it\hat{K}} \chi = \frac{1}{2\pi i} \int_{-\infty + i\mu}^{+\infty + i\mu} e^{-izt} \chi \hat{R}(z) \chi \, dz$$

as operators from $\hat{E}$ to $\hat{E}^{-2} \coloneqq (\hat{K}(s) - \omega)^2 \hat{E}$ ($\Im \omega$ sufficiently large). Then

$$\|\chi \hat{R}(z) \chi u\|_{\hat{E}} \lesssim \langle z \rangle \|\tilde{\chi} p(z; s)^{-1} \tilde{\chi} u\| \quad (\tilde{\chi} \chi = \chi),$$
$$\|\chi \hat{R}_\ell(z) \chi u\|_{\hat{E}^{-2}} \lesssim \langle z \rangle^{-2} \|\chi \hat{R}_\ell(z) \chi u\|_{\hat{E}}$$
Contour deformation

Let $\mu, \nu > 0$. We have

$$\chi e^{-it\hat{K}} \chi = \frac{1}{2\pi i} \int_{-\infty+\mu i}^{+\infty+\mu i} e^{-itz} \chi \hat{R}(z) \chi \, dz$$

as operators from $\hat{E}$ to $\hat{E}^{-2} := (\hat{K}(s) - \omega)^2 \hat{E}$ ($\Im \omega$ sufficiently large). Then

$$\|\chi \hat{R}(z) \chi u\|_{\hat{E}} \lesssim \langle z \rangle \|\hat{\chi} p(z; s)^{-1} \tilde{\chi} u\| \quad (\tilde{\chi} \chi = \chi),$$

$$\|\chi \hat{R}(z) \chi u\|_{\hat{E}^{-2}} \lesssim \langle z \rangle^{-2} \|\chi \hat{R}(z) \chi u\|_{\hat{E}}$$

1. We integrate $e^{-itz} \chi \hat{R}(z) \chi$ over the following contour:
Contour deformation

Let $\mu, \nu > 0$. We have

$$\chi e^{-it\hat{K}} \chi = \frac{1}{2\pi i} \int_{-\infty+i\mu}^{+\infty+i\mu} e^{-izt} \chi \hat{R}(z) \chi \, dz$$

as operators from $\hat{\mathcal{E}}$ to $\hat{\mathcal{E}}^{-2} := (\hat{K}(s) - \omega)^2 \hat{\mathcal{E}}$ ($\Im \omega$ sufficiently large). Then

$$\|\chi \hat{R}(z) \chi u\|_{\hat{\mathcal{E}}} \lesssim \langle z \rangle \|\tilde{\chi} p(z; s)^{-1} \tilde{\chi} u\| \quad (\tilde{\chi} \chi = \chi),$$

$$\|\chi \hat{R}_\ell(z) \chi u\|_{\hat{\mathcal{E}}^{-2}} \lesssim \langle z \rangle^{-2} \|\chi \hat{R}_\ell(z) \chi u\|_{\hat{\mathcal{E}}}$$

2. Letting $K \to +\infty$ and using the above estimates on $\chi \hat{R}_\ell(z) \chi$ and $\chi p_\ell(z, s)^{-1} \chi$ as well as residue theorem gives the expansion.
Asymptotic dynamics

Let $i_{\pm} \in C^\infty(\mathbb{R}, [0, 1])$ as below:

\[ i_- \quad \quad 1 \quad \quad i_+ \]

\[ 0 \quad \quad x \]

Asymptotic equations:

\[ (\partial_t - i_s V \pm) u - \partial^2_x u = 0. \]  

(4)

Positive conserved energies: for $u$ solution of (4),

\[ \langle u | u \rangle_{sV \pm} = \| (\partial_t - i_s V \pm) u \|^2 + \langle -\partial^2_x u, u \rangle > 0. \]

Asymptotic operators and energy spaces:

\[ \hat{H}_{\pm}(s) := (0 \text{Id} - s^2 V^2 \pm 2 s V \pm \{ \text{on } \dot{\mathcal{E}}_{\pm} := C^\infty_c(\mathbb{R} \times S^2) \times C^\infty_c(\mathbb{R} \times S^2) \langle . | . \rangle_{sV \pm} \} \]

\[ \text{For all } \ell \in \mathbb{N}, \text{ set } Z_{\ell} := \frac{1}{\ell (\ell + 1)} (-\Delta_{S^2}) L^2(\mathbb{R} \times S^2, d x) \text{ and let } \mathcal{W}_{\ell} := (Z_{\ell} \otimes L^2(\mathbb{R}, d x)) \oplus \mathcal{E}_{\text{fin}} \pm := \{ u \in \dot{\mathcal{E}}_{\pm} | \exists \ell > 0, u \in \oplus_{\ell \leq \ell} (\dot{\mathcal{E}}_{\pm} \setminus \mathcal{W}_{\ell}) \}. \]
Asymptotic dynamics

Let $i_{\pm} \in C^\infty(\mathbb{R}, [0, 1])$ as below:

Asymptotic equations:

$$(\partial_t - isV_{\pm})^2 u - \partial_x^2 u = 0.$$ (4)
Asymptotic dynamics

Let $i_\pm \in C^\infty(\mathbb{R}, [0, 1])$ as below:

\[
\begin{array}{c}
\includegraphics[width=\textwidth]{asymptotic_dynamics.png}
\end{array}
\]

- Asymptotic equations:
  \[
  (\partial_t - isV_\pm)^2 u - \partial_x^2 u = 0. \tag{4}
  \]

- **Positive** conserved energies: for $u$ solution of (4),
  \[
  \langle u | u \rangle_{sV_\pm} = \|(\partial_t - isV_\pm)u\|^2 + \langle -\partial_x^2 u, u \rangle > 0.
  \]
Asymptotic dynamics

Let \( i_\pm \in C^\infty(\mathbb{R}, [0, 1]) \) as below:

\[
\begin{array}{c}
i_-\\
1\\
i_+
\end{array}
\]

- Asymptotic equations:
  \[
  (\partial_t - isV_\pm)^2 u - \partial_x^2 u = 0. \tag{4}
  \]
- Positive conserved energies: for \( u \) solution of (4),
  \[
  \langle u | u \rangle_{sV_\pm} = \| (\partial_t - isV_\pm)u \|^2 + \langle -\partial_x^2 u, u \rangle > 0.
  \]
- Asymptotic operators and energy spaces:
  \[
  \hat{H}_\pm(s) := \begin{pmatrix} 0 & \operatorname{Id} \\ P - s^2 V_\pm & 2sV_\pm \end{pmatrix} \quad \text{on} \quad \dot{\mathcal{E}}_\pm := C^\infty_c(\mathbb{R} \times S^2) \times C^\infty_c(\mathbb{R} \times S^2)^{\langle \cdot | \cdot \rangle_{sV_\pm}}
  \]
Asymptotic dynamics

Let $i_{\pm} \in C^\infty(\mathbb{R}, [0, 1])$ as below:

\[
\begin{array}{c}
\text{ } \\
\text{i} \\
\text{-} \\
\text{1} \\
\text{0} \\
\text{i} \\
\text{+} \\
\end{array}
\]

\begin{align*}
\text{Asymptotic equations:} & \quad \left( \partial_t - isV_{\pm} \right)^2 u - \partial_x^2 u = 0. \\
\text{Positive conserved energies:} & \quad \langle u | u \rangle_{sV_{\pm}} = \| (\partial_t - isV_{\pm})u \|^2 + \langle -\partial_x^2 u, u \rangle > 0. \\
\text{Asymptotic operators and energy spaces:} & \quad \hat{H}_{\pm}(s) := \begin{pmatrix} 0 & \text{Id} \\ P - s^2 V_{\pm}^2 & 2sV_{\pm} \end{pmatrix} \quad \text{on} \quad \dot{\mathcal{E}}_{\pm} := C_c^\infty(\mathbb{R} \times S^2) \times C_c^\infty(\mathbb{R} \times S^2)_{\langle \cdot, \cdot \rangle_{sV_{\pm}}}
\end{align*}

\begin{align*}
\text{For all } \ell \in \mathbb{N}, \text{ set } Z_\ell := 1_{\ell(\ell+1)}(-\Delta_{S^2})L^2(\mathbb{R} \times S^2, dx d\omega) \text{ and let} \\
W_\ell := (Z_\ell \otimes L^2(\mathbb{R}, dx))^\oplus, \quad \hat{\mathcal{E}}_{\pm}^{\text{fin}} := \{ u \in \dot{\mathcal{E}}_{\pm} | \exists L > 0, u \in \bigoplus_{\ell \leq L} (\dot{\mathcal{E}}_{\pm} \cap W_\ell) \}.
\end{align*}
Asymptotic completeness

Theorem (Georgescu-Gérard-Häfner, 2017)

Assume $s$ sufficiently small.

1. For all $u \in \dot{E}^\text{fin}_\pm$, the limits

$$W_\pm(s)u = \lim_{t \to +\infty} e^{it\hat{H}(s)} i_\pm^2 e^{-it\hat{H}_\pm(s)} u$$

exist in $\dot{E}$. The wave operators $W_\pm$ extend to bounded operators $W_\pm(s) \in B(\dot{E}_\pm, \dot{E})$. 

▶ Same result for the charged KG equation in De Sitter-Kerr spacetime with restricted asymptotic energy spaces in angular directions (restriction of operators and spaces to $\ker(\partial \phi - n)$, $n \in \mathbb{Z}$).

▶ Existence and completeness of wave operators for the wave equation in Kerr spacetime with no angular restriction by Dafermos-Rodnianski-Shlapentokh-Rothman.
Asymptotic completeness

Theorem (Georgescu-Gérard-Häfner, 2017)

Assume $s$ sufficiently small.

1. For all $u \in \dot{\mathcal{E}}^\text{fin}_\pm$, the limits
   \[ W_\pm(s)u = \lim_{t \to +\infty} e^{it\hat{H}(s)}i^2_\pm e^{-it\hat{H}_\pm(s)}u \]
   exist in $\dot{\mathcal{E}}$. The wave operators $W_\pm$ extend to bounded operators $W_\pm(s) \in B(\dot{\mathcal{E}}_\pm, \dot{\mathcal{E}})$.

2. The inverse wave operators
   \[ \Omega_\pm(s) = s - \lim_{t \to +\infty} e^{it\hat{H}_\pm(s)}i^2_\pm e^{-it\hat{H}(s)} \]
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What happens when $s$ becomes \textbf{large}?

- Whiting transformations do not seem to work.
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- Numerical investigation of localization of low frequency resonances in De Sitter-Reissner-Nordström spacetime in progress.
Thank you for your attention!