Boundary value problems on Riemannian and Lorentzian manifolds

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Outline

1. Riemannian manifolds and elliptic operators
   - The Atiyah-Patodi-Singer index theorem
   - General elliptic boundary conditions
2. Lorentzian manifolds and hyperbolic operators
   - Dirac operator on Lorentzian manifolds
   - Fredholm pairs
   - The Lorentzian index theorem
   - More general boundary conditions

1. Riemannian manifolds and elliptic operators

Setup

- M Riemannian manifold, compact, with boundary ∂M
- spin structure $\rightsquigarrow$ spinor bundle $SM \to M$
- $n = \mathrm{dim}(M)$ even $\rightsquigarrow$ splitting $SM = S_R M \oplus S_L M$
  $\rightsquigarrow$ Dirac operator $D : C^\infty(M, S_R M) \to C^\infty(M, S_L M)$

Need boundary conditions:
Let A_0 be the Dirac operator on ∂M.
$P_+ = \chi_{[0,\infty)}(A_0) =$ spectral projector

APS-boundary conditions:

$$P_+(f|_{\partial M}) = 0$$

Atiyah-Patodi-Singer index theorem

Theorem (M. Atiyah, V. Patodi, I. Singer, 1975)

Under APS-boundary conditions D is Fredholm and

$$
\begin{aligned}
\operatorname{ind}(D_{\mathrm{APS}}) = & \int_M \widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E) \\
& + \int_{\partial M} T(\widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E)) - \frac{h(A_0) + \eta(A_0)}{2}
\end{aligned}
$$

Here

- $h(A) = \dim \ker(A)$
- $\eta(A) = \eta_A(0)$ where $\eta_A(s) = \sum_{\substack{\lambda \in \mathrm{spec}(A) \\ \lambda \neq 0}} \mathrm{sign}(\lambda) \cdot |\lambda|^{-s}$

Which boundary conditions other than APS will work?

Warning

APS-boundary conditions cannot be replaced by anti-Atiyah-Patodi-Singer boundary conditions,

$$P_-(f|_{\partial M}) = \chi_{(-\infty,0)}(A_0)(f|_{\partial M}) = 0$$

Example

- M = unit disk $\subset \mathbb{C}$
- $D = \overline{\partial} = \frac{\partial}{\partial \overline{z}}$
- Taylor expansion: $u = \sum_{n=0}^{\infty} \alpha_n z^n$
- $A_0 = i\frac{d}{d\theta}$
- Fourier expansion: $u|_{\partial M} = \sum_{n\in\mathbb{Z}} \alpha_n e^{in\theta}$

APS-boundary conditions:
$\alpha_n = 0$ for $n \geq 0 \Rightarrow \ker(D) = \{0\}$
aAPS-boundary conditions:
$\alpha_n = 0$ for $n < 0 \Rightarrow \ker(D) =$ infinite dimensional

Generalize APS conditions

Notation

For an interval $J \subset \mathbb{R}$ write

$$L^2_J(\partial M) = \left\{ u \in L^2(\partial M) \,\middle|\, u = \sum_{\lambda \in J \cap \mathrm{spec}(A_0)} a_\lambda \varphi_\lambda \right\}$$

where $A_0 \varphi_\lambda = \lambda \varphi_\lambda$. Similarly for $H^s_J(\partial M)$.

APS-boundary conditions

$$f|_{\partial M} \in B = H^{\frac{1}{2}}_{(-\infty,0)}(\partial M)$$

1. Generalization

Replace $(-\infty, 0)$ by $(-\infty, a]$ for some $a \in \mathbb{R}$:

$$B = H^{\frac{1}{2}}_{(-\infty,a]}(\partial M)$$

Generalize APS conditions

2. Generalization

Deform

$$B = \{v + gv \mid v \in H^{\frac{1}{2}}_{(-\infty,a]}(\partial M)\}$$

where $g : H^{\frac{1}{2}}_{(-\infty,a]}(\partial M) \to H^{\frac{1}{2}}_{(a,\infty)}(\partial M)$ is bounded linear.

3. Generalization

Finite-dimensional modification

$$B = W_+ \oplus \{v + gv \mid v \in H^{\frac{1}{2}}_{(-\infty,a]}(\partial M)\}$$

where $W_+ \subset C^\infty(\partial M)$ is finite-dimensional.

Elliptic boundary conditions

Definition

A linear subspace $B \subset H^{\frac{1}{2}}(\partial M)$ is said to be an elliptic boundary condition if there is an L^2-orthogonal decomposition

$$L^2(\partial M) = V_- \oplus W_- \oplus V_+ \oplus W_+$$

such that

$$B = W_+ \oplus \{v + gv \mid v \in V_- \cap H^{\frac{1}{2}}\}$$

where

1) $W_\pm \subset C^\infty(\partial M)$ finite-dimensional;
2) $V_- \oplus W_- \subset L^2_{(-\infty,a]}(\partial M)$ and $V_+ \oplus W_+ \subset L^2_{[-a,\infty)}(\partial M)$, for some $a \in \mathbb{R}$;
3) $g : V_- \to V_+$ and $g^* : V_+ \to V_-$ are operators of order 0.

Fredholm property and boundary regularity

Theorem (Ballmann-B. 2012)

Let B be an elliptic boundary condition. Then

$$D_B : \{f \in H^1(M, V_R) \mid f|_{\partial M} \in B\} \to L^2(M, V_L)$$

is Fredholm.

Theorem (Ballmann-B. 2012)

Let B be an elliptic boundary condition. Then

$$f \in H^{k+1}(M, V_R) \Longleftrightarrow D_B f \in H^k(M, V_L),$$

for all $f \in \operatorname{dom} D_B$ and $k \geq 0$.
In particular, $f \in \operatorname{dom} D_B$ is smooth up to the boundary iff $D_B f$ is smooth up to the boundary.

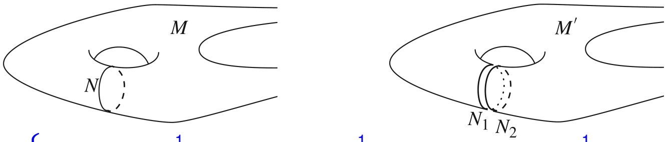
Examples

1) Generalized APS:
$V_- = L^2_{(-\infty,a)}(A_0)$, $V_+ = L^2_{[a,\infty)}(A_0)$, $W_- = W_+ = \{0\}$, $g = 0$.
Then
$$B = H^{\frac{1}{2}}_{(-\infty,a)}(A_0).$$

2) Classical local elliptic boundary conditions in the sense of Lopatinsky-Schapiro.

Examples

3) "Transmission" condition



$$B = \left\{ (\phi, \phi) \in H^{\frac{1}{2}}(N_1, V_R) \oplus H^{\frac{1}{2}}(N_2, V_R) \mid \phi \in H^{\frac{1}{2}}(N, V_R) \right\}$$

Here

$$V_+ = L^2_{(0,\infty)}(A_0 \oplus -A_0) = L^2_{(0,\infty)}(A_0) \oplus L^2_{(-\infty,0)}(A_0)$$

$$V_- = L^2_{(-\infty,0)}(A_0 \oplus -A_0) = L^2_{(-\infty,0)}(A_0) \oplus L^2_{(0,\infty)}(A_0)$$

$$W_+ = \{(\phi, \phi) \in \ker(A_0) \oplus \ker(A_0)\}$$

$$W_- = \{(\phi, -\phi) \in \ker(A_0) \oplus \ker(A_0)\}$$

$$g : V_- \to V_+, \quad g = \begin{pmatrix} 0 & \mathrm{id} \\ \mathrm{id} & 0 \end{pmatrix}$$

A deformation argument

Replace B by B_s where g is replaced by g_s with $g_s = s \cdot g$.
Then $B_1 =$ transmission condition and $B_0 =$ APS-condition.

Hence $\mathrm{ind}(D^M) = \mathrm{ind}(D^{M'}_{transm.}) = \mathrm{ind}(D^{M'}_{\mathrm{APS}})$.

Holds also if M is complete noncompact and D satisfies a coercivity condition at infinity.

Implies **relative index theorem** by Gromov and Lawson (1983).

2. Lorentzian manifolds and hyperbolic operators

Index theory in Lorentzian signature?

Problem 1: Let D be a differential operator of order k over a closed manifold. Then $D : H^k \to L^2$ is Fredholm $\Leftrightarrow$ D is elliptic.

$\Rightarrow$ no Lorentzian analog to Atiyah-Singer index theorem

Problem 2: Hyperbolic PDEs behave badly on closed manifolds

Problem 3: Closed Lorentzian manifolds violate causality conditions

$\Rightarrow$ useless as models in General Relativity

But: There exists a Lorentzian analog to the Atiyah-Patodi-Singer index theorem!

Globally hyperbolic spacetimes

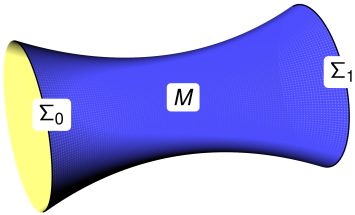
A subset $\Sigma \subset M$ is called Cauchy hypersurface if each inextendible timelike curve in M meets Σ exactly once.

If M has a Cauchy hypersurface then M is called globally hyperbolic.

Examples:

- Minkowski spacetime (Special Relativity)
- Schwarzschild Model (Black Hole)
- Friedmann cosmos (Big Bang, cosmic expansion)
- deSitter spacetime
- $\cdots$

Globally hyperbolic spacetimes



Let M be a globally hyperbolic Lorentzian manifold **with boundary** $\partial M = \Sigma_0 \sqcup \Sigma_1$

Σ_j compact smooth spacelike Cauchy hypersurfaces

The Cauchy problem

Well-posedness of Cauchy problem

The map $D \oplus \mathrm{res}_\Sigma : C^\infty(M; V_R) \to C^\infty(M; V_L) \oplus C^\infty(\Sigma; V_R)$ is an isomorphism of topological vector spaces.

Wave propagator U:

$$
\begin{tikzcd}
& \{v \in C^\infty(M; V_R) \mid Dv = 0\} \arrow[dl, "\mathrm{res}_{\Sigma_0}"', "\cong"] \arrow[dr, "\mathrm{res}_{\Sigma_1}", "\cong"'] & \\
C^\infty(\Sigma_0, V_R) \arrow[rr, "U"] & & C^\infty(\Sigma_1, V_R)
\end{tikzcd}
$$

U extends to **unitary** operator $L^2(\Sigma_0; V_R) \to L^2(\Sigma_1; V_R)$.

Fredholm pairs

Definition

Let H be a Hilbert space and $B_0, B_1 \subset H$ closed linear subspaces. Then (B_0, B_1) is called a Fredholm pair if $B_0 \cap B_1$ is finite dimensional and $B_0 + B_1$ is closed and has finite codimension. The number

$$\operatorname{ind}(B_0, B_1) = \dim(B_0 \cap B_1) - \dim(H/(B_0 + B_1))$$

is called the index of the pair (B_0, B_1).

Elementary properties:

1.) $\operatorname{ind}(B_0, B_1) = \operatorname{ind}(B_1, B_0)$

2.) $\operatorname{ind}(B_0, B_1) = -\operatorname{ind}(B_0^\perp, B_1^\perp)$

3.) Let $B_0 \subset B_0'$ with $\dim(B_0'/B_0) < \infty$. Then

$$\operatorname{ind}(B_0', B_1) = \operatorname{ind}(B_0, B_1) + \dim(B_0'/B_0).$$

Fredholm pairs and the Dirac operator

Let $B_0 \subset L^2(\Sigma_0, V_R)$ and $B_1 \subset L^2(\Sigma_1, V_R)$ be closed subspaces.

Proposition (B.-Hannes 2017)

The following are equivalent:

(i) The pair $(B_0, U^{-1}B_1)$ is Fredholm of index k;

(ii) The pair (UB_0, B_1) is Fredholm of index k;

(iii) The restriction

$$D : \{f \in FE(M, V_R) \mid f|_{\Sigma_i} \in B_i\} \to L^2(M, V_L)$$

is a Fredholm operator of index k.

Trivial example

Let $\mathrm{dim}(B_0) < \infty$ and $\mathrm{codim}(B_1) < \infty$.

Then D with these boundary conditions is Fredholm with index

$$\mathrm{dim}(B_0) - \mathrm{codim}(B_1)$$

The Lorentzian index theorem

Theorem (B.-Strohmaier 2015)

Under APS-boundary conditions D is Fredholm.
The kernel consists of smooth spinor fields and

$$\mathrm{ind}(D_{\mathrm{APS}}) = \int_M \widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E) + \int_{\partial M} T(\widehat{\mathrm{A}}(M) \wedge \mathrm{ch}(E)) - \frac{h(A_0) + h(A_1) + \eta(A_0) - \eta(A_1)}{2}$$

$$\begin{aligned}\mathrm{ind}(D_{\mathrm{APS}}) = &\dim\ker[D : C^\infty_{\mathrm{APS}}(M; V_R) \to C^\infty(M; V_L)] \\ &- \dim\ker[D : C^\infty_{\mathrm{aAPS}}(M; V_R) \to C^\infty(M; V_L)]\end{aligned}$$

aAPS conditions are as good as APS-boundary conditions.

Boundary conditions in graph form

A pair (B_0, B_1) of closed subspaces $B_i \subset L^2(\Sigma_i, V_R)$ form elliptic boundary conditions if there are L^2-orthogonal decompositions

$$L^2(\Sigma_i, V_R) = V_i^- \oplus W_i^- \oplus V_i^+ \oplus W_i^+, \qquad i = 0, 1,$$

such that

(i) W_i^+, W_i^- are finite dimensional;

(ii) $W_i^- \oplus V_i^- \subset L^2_{(-\infty, a_i]}(\partial M)$ and $W_i^+ \oplus V_i^+ \subset L^2_{[-a_i, \infty)}(\partial M)$ for some $a_i \in \mathbb{R}$;

(iii) There are bounded linear maps $g_0 : V_0^- \to V_0^+$ and $g_1 : V_1^+ \to V_1^-$ such that

$$B_0 = W_0^+ \oplus \Gamma(g_0),$$
$$B_1 = W_1^- \oplus \Gamma(g_1),$$

where $\Gamma(g_{0/1}) = \{v + g_{0/1} v \mid v \in V_{0/1}^{\mp}\}$.

Boundary conditions in graph form

Theorem (B.-Hannes 2017)

The pair (B_0, B_1) is Fredholm provided

(A) g_0 or g_1 is compact **or**

(B) $\|g_0\| \cdot \|g_1\|$ is small enough.

1.) Applies if $g_0 = 0$ or $g_1 = 0$.

2.) Conditions (A) and (B) cannot both be dropped (counterexamples).

Counterexample

Put $M = [0,1] \times S^1$ with $g = -dt^2 + d\theta^2$. Then

$$U = \mathrm{id} : L^2(\Sigma_0) = L^2(S^1) \to L^2(\Sigma_1) = L^2(S^1)$$

Now choose

$$\begin{aligned} &V_0^- = L^2_{(-\infty,0)}(A), \quad V_0^+ = L^2_{(0,\infty)}(A), \quad W_0^- = \ker(A), \quad W_0^+ = 0, \\ &V_1^- = L^2_{(-\infty,0)}(A), \quad V_1^+ = L^2_{(0,\infty)}(A), \quad W_1^- = 0, \quad W_1^+ = \ker(A). \end{aligned}$$

Let $g_0 : L^2_{(-\infty,0)}(A) \to L^2_{(0,\infty)}(A)$ be unitary and put $g_1 = g_0^{-1}$.
Then

$$B_0 = \Gamma(g_0) = \{v + g_0 v \mid v \in L^2_{(-\infty,0)}(A)\}$$

$$B_1 = \Gamma(g_1) = \{g_1 w + w \mid w \in L^2_{(0,\infty)}(A)\}.$$

Now $(B_0, U^{-1}B_1) = (B_0, B_1) = (B_0, B_0)$ is not a Fredholm pair.

Summary

	Riemannian	Lorentzian
APS	$\checkmark$	$\checkmark$
aAPS	-	$\checkmark$
elliptic b.c.	$\checkmark$	depends

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