Boundary value problems on Riemannian and Lorentzian manifolds

Christian Bär
(joint with W. Ballmann, S. Hannes, A. Strohmaier)

Institut für Mathematik
Universität Potsdam

Quantum fields, scattering and spacetime horizons: mathematical challenges

Les Houches, May 22, 2018
1. Riemannian manifolds and elliptic operators
   - The Atiyah-Patodi-Singer index theorem
   - General elliptic boundary conditions

2. Lorentzian manifolds and hyperbolic operators
   - Dirac operator on Lorentzian manifolds
   - Fredholm pairs
   - The Lorentzian index theorem
   - More general boundary conditions
1. Riemannian manifolds and elliptic operators
Setup

- $M$ Riemannian manifold, compact, with boundary $\partial M$
- spin structure $\leadsto$ spinor bundle $SM \to M$
- $n = \dim(M)$ even $\leadsto$ splitting $SM = S_R M \oplus S_L M$
  $\leadsto$ Dirac operator $D : C^\infty(M, S_R M) \to C^\infty(M, S_L M)$

Need boundary conditions:
Let $A_0$ be the Dirac operator on $\partial M$.
$P_+ = \chi_{[0,\infty)}(A_0) =$ spectral projector

APS-boundary conditions:

$$P_+(f|_{\partial M}) = 0$$
Atiyah-Patodi-Singer index theorem

Theorem (M. Atiyah, V. Patodi, I. Singer, 1975)

Under APS-boundary conditions $D$ is Fredholm and

$$\text{ind}(D_{\text{APS}}) = \int_M \hat{A}(M) \wedge \text{ch}(E)$$

$$+ \int_{\partial M} T(\hat{A}(M) \wedge \text{ch}(E)) - \frac{h(A_0) + \eta(A_0)}{2}$$

Here

- $h(A) = \dim \ker(A)$
- $\eta(A) = \eta_A(0)$ where $\eta_A(s) = \sum_{\lambda \in \text{spec}(A)} \text{sign}(\lambda) \cdot |\lambda|^{-s}$
Which boundary conditions other than APS will work?
Warning

APS-boundary conditions cannot be replaced by anti-Atiyah-Patodi-Singer boundary conditions,

\[ P_-(f|_{\partial M}) = \chi_{(-\infty,0)}(A_0)(f|_{\partial M}) = 0 \]

Example

- \( M = \) unit disk \( \subset \mathbb{C} \)
- \( D = \overline{\partial} = \frac{\partial}{\partial \overline{z}} \)
- Taylor expansion: \( u = \sum_{n=0}^{\infty} \alpha_n z^n \)
- \( A_0 = i \frac{d}{d\theta} \)
- Fourier expansion: \( u|_{\partial M} = \sum_{n \in \mathbb{Z}} \alpha_n e^{in\theta} \)

APS-boundary conditions:
\( \alpha_n = 0 \) for \( n \geq 0 \) \( \Rightarrow \) \( \ker(D) = \{0\} \)
aAPS-boundary conditions:
\( \alpha_n = 0 \) for \( n < 0 \) \( \Rightarrow \) \( \ker(D) = \) infinite dimensional
Generalize APS conditions

Notation

For an interval $J \subset \mathbb{R}$ write

$$L^2_J(\partial M) = \left\{ u \in L^2(\partial M) \mid u = \sum_{\lambda \in J \cap \text{spec}(A_0)} a_\lambda \varphi_\lambda \right\}$$

where $A_0 \varphi_\lambda = \lambda \varphi_\lambda$. Similarly for $H^s_J(\partial M)$.

APS-boundary conditions

$$f|_{\partial M} \in B = H^1_{(-\infty,0)}(\partial M)$$

1. Generalization

Replace $(-\infty,0)$ by $(-\infty,a]$ for some $a \in \mathbb{R}$:

$$B = H^1_{(-\infty,a]}(\partial M)$$
2. Generalization

Deform

\[ B = \{ v + g v \mid v \in H^1_{(\infty, a]}(\partial M) \} \]

where \( g : H^1_{(-\infty, a]}(\partial M) \rightarrow H^1_{(a, \infty)}(\partial M) \) is bounded linear.

3. Generalization

Finite-dimensional modification

\[ B = W_+ \oplus \{ v + g v \mid v \in H^1_{(-\infty, a]}(\partial M) \} \]

where \( W_+ \subset C^\infty(\partial M) \) is finite-dimensional.
Elliptic boundary conditions

Definition

A linear subspace $B \subset H^\frac{1}{2}(\partial M)$ is said to be an elliptic boundary condition if there is an $L^2$-orthogonal decomposition

$$L^2(\partial M) = V_- \oplus W_- \oplus V_+ \oplus W_+$$

such that

$$B = W_+ \oplus \{v + gv \mid v \in V_- \cap H^\frac{1}{2}\}$$

where

1) $W_\pm \subset C^\infty(\partial M)$ finite-dimensional;

2) $V_- \oplus W_- \subset L^2_{(-\infty,a]}(\partial M)$ and $V_+ \oplus W_+ \subset L^2_{[-a,\infty)}(\partial M)$, for some $a \in \mathbb{R};$

3) $g : V_- \to V_+$ and $g^* : V_+ \to V_-$ are operators of order 0.
Fredholm property and boundary regularity

Theorem (Ballmann-B. 2012)

Let $B$ be an elliptic boundary condition. Then

$$D_B : \{ f \in H^1(M, V_R) \mid f|_{\partial M} \in B \} \to L^2(M, V_L)$$

is Fredholm.

Theorem (Ballmann-B. 2012)

Let $B$ be an elliptic boundary condition. Then

$$f \in H^{k+1}(M, V_R) \iff D_B f \in H^k(M, V_L),$$

for all $f \in \text{dom } D_B$ and $k \geq 0$. In particular, $f \in \text{dom } D_B$ is smooth up to the boundary iff $D_B f$ is smooth up to the boundary.
Examples

1) Generalized APS:
\[ V_- = L^2_{(-\infty,a)}(A_0), \quad V_+ = L^2_{[a,\infty)}(A_0), \quad W_- = W_+ = \{0\}, \quad g = 0. \]
Then
\[ B = H^1_{(-\infty,a)}(A_0). \]

2) Classical local elliptic boundary conditions in the sense of Lopatinsky-Schapiro.
3) “Transmission” condition

\[
B = \left\{ (\phi, \phi) \in H^1_2(N_1, V_R) \oplus H^1_2(N_2, V_R) \mid \phi \in H^2_2(N, V_R) \right\}
\]

Here

\[
V_+ = L^2_{(0,\infty)}(A_0 \oplus -A_0) = L^2_{(0,\infty)}(A_0) \oplus L^2_{(-\infty,0)}(A_0)
\]
\[
V_- = L^2_{(-\infty,0)}(A_0 \oplus -A_0) = L^2_{(-\infty,0)}(A_0) \oplus L^2_{(0,\infty)}(A_0)
\]
\[
W_+ = \left\{ (\phi, \phi) \in \ker(A_0) \oplus \ker(A_0) \right\}
\]
\[
W_- = \left\{ (\phi, -\phi) \in \ker(A_0) \oplus \ker(A_0) \right\}
\]
\[
g : V_- \to V_+, \quad g = \begin{pmatrix} 0 & \text{id} \\ \text{id} & 0 \end{pmatrix}
\]
A deformation argument

Replace $B$ by $B_s$ where $g$ is replaced by $g_s$ with $g_s = s \cdot g$. Then $B_1 =$ transmission condition and $B_0 =$ APS-condition.

Hence $\text{ind}(D^M) = \text{ind}(D^M_{\text{transm.}}) = \text{ind}(D^M_{\text{APS}})$.

Holds also if $M$ is complete noncompact and $D$ satisfies a coercivity condition at infinity.

Implies relative index theorem by Gromov and Lawson (1983).
2. Lorentzian manifolds and hyperbolic operators
Index theory in Lorentzian signature?

**Problem 1**: Let $D$ be a differential operator of order $k$ over a closed manifold. Then $D : H^k \to L^2$ is Fredholm $\iff$ $D$ is elliptic.

$\Rightarrow$ no Lorentzian analog to Atiyah-Singer index theorem

**Problem 2**: Hyperbolic PDEs behave badly on closed manifolds

**Problem 3**: Closed Lorentzian manifolds violate causality conditions

$\Rightarrow$ useless as models in General Relativity

**But**: There exists a Lorentzian analog to the Atiyah-Patodi-Singer index theorem!
A subset $\Sigma \subset M$ is called **Cauchy hypersurface** if each inextendible timelike curve in $M$ meets $\Sigma$ exactly once.

If $M$ has a Cauchy hypersurface then $M$ is called **globally hyperbolic**.

**Examples:**

- Minkowski spacetime (Special Relativity)
- Schwarzschild Model (Black Hole)
- Friedmann cosmos (Big Bang, cosmic expansion)
- deSitter spacetime
- …
Let $M$ be a globally hyperbolic Lorentzian manifold with boundary $\partial M = \Sigma_0 \sqcup \Sigma_1$ and $\Sigma_j$ compact smooth spacelike Cauchy hypersurfaces.
The Cauchy problem

Well-posedness of Cauchy problem

The map \( D \oplus \text{res}_\Sigma : C^\infty(M; V_R) \to C^\infty(M; V_L) \oplus C^\infty(\Sigma; V_R) \) is an isomorphism of topological vector spaces.

Wave propagator \( U \):

\[
\begin{align*}
\{ v \in C^\infty(M; V_R) \mid Dv = 0 \} & \xrightarrow{\text{res}_\Sigma_0} C^\infty(\Sigma_0, V_R) \\
& \xrightarrow{U} C^\infty(\Sigma_1, V_R) \\
& \xleftarrow{\text{res}_\Sigma_1} L^2(\Sigma_0; V_R)
\end{align*}
\]

\( U \) extends to unitary operator \( L^2(\Sigma_0; V_R) \to L^2(\Sigma_1; V_R) \).
Fredholm pairs

Definition

Let $H$ be a Hilbert space and $B_0, B_1 \subset H$ closed linear subspaces. Then $(B_0, B_1)$ is called a Fredholm pair if $B_0 \cap B_1$ is finite dimensional and $B_0 + B_1$ is closed and has finite codimension. The number

$$\text{ind}(B_0, B_1) = \dim(B_0 \cap B_1) - \dim(H/(B_0 + B_1))$$

is called the index of the pair $(B_0, B_1)$.

Elementary properties:

1.) $\text{ind}(B_0, B_1) = \text{ind}(B_1, B_0)$
2.) $\text{ind}(B_0, B_1) = -\text{ind}(B_0^\perp, B_1^\perp)$
3.) Let $B_0 \subset B'_0$ with $\dim(B'_0/B_0) < \infty$. Then

$$\text{ind}(B'_0, B_1) = \text{ind}(B_0, B_1) + \dim(B'_0/B_0).$$
Fredholm pairs and the Dirac operator

Let $B_0 \subset L^2(\Sigma_0, V_R)$ and $B_1 \subset L^2(\Sigma_1, V_R)$ be closed subspaces.

**Proposition (B.-Hannes 2017)**

The following are equivalent:

(i) The pair $(B_0, U^{-1}B_1)$ is Fredholm of index $k$;

(ii) The pair $(UB_0, B_1)$ is Fredholm of index $k$;

(iii) The restriction

$$D : \{ f \in FE(M, V_R) \mid f|_{\Sigma_i} \in B_i \} \to L^2(M, V_L)$$

is a Fredholm operator of index $k$. 
Let $\dim(B_0) < \infty$ and $\text{codim}(B_1) < \infty$.

Then $D$ with these boundary conditions is Fredholm with index

$$\dim(B_0) - \text{codim}(B_1)$$
The Lorentzian index theorem

Theorem (B.-Strohmaier 2015)

Under APS-boundary conditions $D$ is Fredholm. The kernel consists of smooth spinor fields and

$$\text{ind}(D_{\text{APS}}) = \int_M \hat{A}(M) \wedge \text{ch}(E) + \int_{\partial M} T(\hat{A}(M) \wedge \text{ch}(E))$$

$$- \frac{h(A_0) + h(A_1) + \eta(A_0) - \eta(A_1)}{2}$$

$$\text{ind}(D_{\text{APS}}) = \dim \ker[D : C^\infty_{\text{APS}}(M; V_R) \to C^\infty(M; V_L)]$$

$$- \dim \ker[D : C^\infty_{\text{aAPS}}(M; V_R) \to C^\infty(M; V_L)]$$

aAPS conditions are as good as APS-boundary conditions.
A pair \((B_0, B_1)\) of closed subspaces \(B_i \subset L^2(\Sigma_i, V_R)\) form \textit{elliptic boundary conditions} if there are \(L^2\)-orthogonal decompositions

\[
L^2(\Sigma_i, V_R) = V_i^- \oplus W_i^- \oplus V_i^+ \oplus W_i^+, \quad i = 0, 1,
\]

such that

(i) \(W_i^+, W_i^-\) are finite dimensional;

(ii) \(W_i^- \oplus V_i^- \subset L^2(-\infty,a_i](\partial M)\) and \(W_i^+ \oplus V_i^+ \subset L^2(-a_i,\infty)(\partial M)\) for some \(a_i \in \mathbb{R}\);

(iii) There are bounded linear maps \(g_0 : V_0^- \to V_0^+\) and \(g_1 : V_1^+ \to V_1^-\) such that

\[
B_0 = W_0^+ \oplus \Gamma(g_0),
\]

\[
B_1 = W_1^- \oplus \Gamma(g_1),
\]

where \(\Gamma(g_{0/1}) = \{v + g_{0/1}v \mid v \in V_{0/1}^\mp\}\).
Theorem (B.-Hannes 2017)

The pair \((B_0, B_1)\) is Fredholm provided

(A) \(g_0\) or \(g_1\) is compact \textbf{or}

(B) \(\|g_0\| \cdot \|g_1\|\) is small enough.

1.) Applies if \(g_0 = 0\) or \(g_1 = 0\).

2.) Conditions (A) and (B) cannot both be dropped (counterexamples).
Counterexample

Put $M = [0, 1] \times S^1$ with $g = -dt^2 + d\theta^2$. Then

$$U = \text{id} : L^2(\Sigma_0) = L^2(S^1) \to L^2(\Sigma_1) = L^2(S^1)$$

Now choose

\[
\begin{align*}
V_0^- &= L^2_{(-\infty,0)}(A), & V_0^+ &= L^2_{(0,\infty)}(A), & W_0^- &= \ker(A), & W_0^+ &= 0, \\
V_1^- &= L^2_{(-\infty,0)}(A), & V_1^+ &= L^2_{(0,\infty)}(A), & W_1^- &= 0, & W_1^+ &= \ker(A).
\end{align*}
\]

Let $g_0 : L^2_{(-\infty,0)}(A) \to L^2_{(0,\infty)}(A)$ be unitary and put $g_1 = g_0^{-1}$. Then

\[
\begin{align*}
B_0 &= \Gamma(g_0) = \{ v + g_0 v \mid v \in L^2_{(-\infty,0)}(A) \} \\
B_1 &= \Gamma(g_1) = \{ g_1 w + w \mid w \in L^2_{(0,\infty)}(A) \}.
\end{align*}
\]

Now $(B_0, U^{-1}B_1) = (B_0, B_1) = (B_0, B_0)$ is not a Fredholm pair.
## Summary

<table>
<thead>
<tr>
<th></th>
<th>Riemannian</th>
<th>Lorentzian</th>
</tr>
</thead>
<tbody>
<tr>
<td>APS</td>
<td>✓</td>
<td>✓</td>
</tr>
<tr>
<td>aAPS</td>
<td>-</td>
<td>✓</td>
</tr>
<tr>
<td>elliptic b.c.</td>
<td>✓</td>
<td>depends</td>
</tr>
</tbody>
</table>
• C. Bär and W. Ballmann: *Boundary value problems for elliptic differential operators of first order*
  arXiv:1101.1196

  arXiv:1506.00959

• C. Bär and A. Strohmaier: *A rigorous geometric derivation of the chiral anomaly in curved backgrounds*
  arXiv:1508.05345

• C. Bär and S. Hannes: *Boundary value problems for the Lorentzian Dirac operator*
  arXiv:1704.03224